Horses and Rabbits? Trade-Off Theory and Optimal Capital Structure

Appendix A

In this appendix we develop the dynamic model in more detail. To obtain $G(T), H(T),$ and $I(T)$[1] in equations (4), (5), and (6) of the main text, we require the first passage time density function. To compute it, we first define

$$x(t) \equiv \log\left(\frac{V(t)}{F_L e^{-g(T-t)}}\right). \tag{A.1}$$

One application of Ito's lemma under the risk-neutral measure yields

$$dx = \left(r - \delta - g - \sigma^2/2\right)dt + \sigma\, dZ^Q(t). \tag{A.2}$$

Consequently, $x(t)$ is a Brownian motion with drift $m \equiv r - \delta - g - \sigma^2/2$ and diffusion σ, starting at $x_0 = \log\left(\frac{V(0)}{F_L e^{-gT}}\right)$. From Ingersoll (1987), the first-passage time density function $f(t)$ for crossing the origin is given by

$$f(t) = \frac{x_0}{\sigma t^{3/2}} n\left(\frac{x_0 + mt}{\sigma t^{1/2}}\right), \tag{A.3}$$

where $n(\bullet)$ is the standard normal density function. Now, lengthy, but straightforward calculations yield[2]

$$G(T) = N\left[h_1(T)\right] + \left(\frac{V(0)}{F_L e^{-gT}}\right)^{-2a} N\left[h_2(T)\right], \tag{A.4}$$

$$H(T) = \left(\frac{V(0)}{F_L e^{-gT}}\right)^{-a+z} N\left[q_1(T)\right] + \left(\frac{V(0)}{F_L e^{-gT}}\right)^{-a-z} N\left[q_2(T)\right], \tag{A.5}$$

$$I(T) = \left(\frac{V(0)}{F_L e^{-gT}}\right)^{-a+\bar{z}} N\left[\bar{q}_1(T)\right] + \left(\frac{V(0)}{F_L e^{-gT}}\right)^{-a-\bar{z}} N\left[\bar{q}_2(T)\right], \tag{A.6}$$

where

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[1] For simplicity, we omit the other arguments of these functions.

[2] Explicit derivation is available upon request.

$$h_1(T) \equiv \left(\frac{-x_0 - a\sigma^2 T}{\sigma\sqrt{T}} \right), \qquad h_2(T) \equiv \left(\frac{-x_0 + a\sigma^2 T}{\sigma\sqrt{T}} \right),$$

$$q_1(T) \equiv \left(\frac{-x_0 - z\sigma^2 T}{\sigma\sqrt{T}} \right), \qquad q_2(T) \equiv \left(\frac{-x_0 + z\sigma^2 T}{\sigma\sqrt{T}} \right), \tag{A.7}$$

$$\bar{q}_1(T) \equiv \left(\frac{-x_0 - \bar{z}\sigma^2 T}{\sigma\sqrt{T}} \right), \qquad \bar{q}_2(T) \equiv \left(\frac{-x_0 + \bar{z}\sigma^2 T}{\sigma\sqrt{T}} \right),$$

$$a \equiv \frac{\left(r - \delta - g - \sigma^2/2\right)}{\sigma^2}, \qquad z \equiv \frac{\left[\left(a\sigma^2\right)^2 + 2r\sigma^2\right]^{1/2}}{\sigma^2}, \qquad \bar{z} \equiv \frac{\left[\left(a\sigma^2\right)^2 + 2\left(r - g\right)\sigma^2\right]^{1/2}}{\sigma^2}.$$

In these expressions, $N(\bullet)$ is the cumulative standard normal distribution function.

Given $G\left(T\right), H\left(T\right),$ and $I\left(T\right),$ the values of the debt, bankruptcy costs, and tax shields of the current debt are given by equations (8), (11), and (13) respectively. The total firm value in the static model, when debt is issued only once by the firm, is given by the value of the firm's unlevered assets plus the tax shields of debt (equation (13)) minus the bankruptcy costs (equation (11))

$$TV_L(0) = V\left(0\right) + TB_L\left(0\right) - BC_L\left(0\right) \ . \tag{A.8}$$

We now turn our attention to the dynamic model. In this model the firm repeatedly and optimally issues *T*-year maturity debt until it goes bankrupt. Obviously, the optimal coupon for the new issues will depend on the firm value when the future debt is issued. We note, however, the following scaling property: If the optimal coupon of the first (initial) debt issue is C_L, then the optimal coupon in future issues will be scaled by the ratio of the asset value $V\left(t\right)$ when the new debt is issued to the initial asset value $V\left(0\right)$. The reason for this is that at time t the firm is identical to itself at time zero, except that it is $V\left(t\right)/V\left(0\right)$ as large because the asset value follows a proportional process (geometric Brownian motion). Therefore, if no bankruptcy has occurred by the time that the initial debt matures at T, the optimal coupon of the new debt will be $C_L\left(V(T)/V(0)\right)$. Now, if bankruptcy occurs at $t^*<T$, the asset value will be $F_L e^{-g\left(T-t^*\right)}$. We allow the debtholders to become the new shareholders, and they optimally lever the remaining asset value (1-α_{BC}) $F_L e^{-g\left(T-t^*\right)}$ after the bankruptcy process consumes α_{BC} $F_L e^{-g\left(T-t^*\right)}$. Thus, the optimal coupon after bankruptcy reorganization is $C_L\,\frac{(1-\alpha_{BC})F_L e^{-g(T-t^*)}}{V(0)}$. In fact, all future issues of debt will be scaled by the ratio of the firm's asset value when the new debt is issued to its asset value when the old debt is issued.

Even though only the current issue of debt is outstanding at time zero, the tax shields and bankruptcy costs reflect all expected future debt issues. Let $TB_L^{Dynamic}(0)$ denote the total tax shields in the dynamic model. The scaling property discussed in the previous paragraph implies that the total tax shields at time T will be $TB_L^{Dynamic}(0)\left(V(T)/V(0)\right)$ if no bankruptcy has yet occurred and it will be $TB_L^{Dynamics}(0)\frac{(1-\alpha_{BC})F_L e^{-g(T-t^*)}}{V(0)}$ if bankruptcy occurs at t^*. Risk-neutral valuation implies that the initial *total* tax shield, $TB_L^{Dynamic}(0)$, is equal to the tax shield from the initial debt plus the conditional discounted risk-neutral expected total tax shield at time T plus the conditional discounted risk-neutral expected total tax shield if bankruptcy occurs at $t^*<T$. If we now let $TB_L(0)$ denote the tax shield from the initial issue of debt, we have[3]

$$\begin{aligned} TB_L^{Dynamic}(0) &= TB_L(0)+\int_{F_L}^{\infty} e^{-rT}\,\frac{V(T)}{V(0)}TB_L^{Dynamic}(0)\,\rho\left(V(T)\right)dV(T) \\ &\quad+\int_0^T e^{-rt^*}\,\frac{(1-\alpha)F_L e^{-g(T-t^*)}}{V(0)}TB_L^{Dynamic}(0)\,f(t^*)\,dt^* \\ &= TB_L(0)+\phi TB_L^{Dynamic}(0), \end{aligned} \tag{A.9}$$

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[3] For simplicity, we omit the other arguments of $\rho\left(V(T)\right)$ and $f(t^*)$.

where ϕ is defined in equation (15). Solving for $TB_L^{Dynamic}(0)$, we obtain

$$TB_L^{Dynamic}(0) = \frac{TB_L(0)}{1-\phi}. \tag{A.10}$$

The total tax shields have an intuitive series expansion. Each term in the expansion

$$TB_L^{Dynamic}(0) = TB_L(0)\left(1+\phi+\phi^2+\phi^3+\cdots\right) \tag{A.11}$$

represents the present value of the tax shields from the debt issue in each succeeding period.

To find ϕ, we require the conditional distribution of $V(T)$ such that the firm has not gone bankrupt at time T. Again, from Ingersoll (1987) we have the following conditional density function for $V(T)$:

$$\rho(V(T)) = \frac{1}{V(T)\sigma\sqrt{T}} n\left(\frac{x(T)-x_0-mT}{\sigma\sqrt{T}}\right) - \frac{e^{\frac{2mx_0}{\sigma^2}}}{V(T)\sigma\sqrt{T}} n\left(\frac{x(T)+x_0-mT}{\sigma\sqrt{T}}\right). \tag{A.12}$$

Using the above density function and the first passage time density $f(t^*)$ given in equation (A.3), tedious but straightforward derivations yield the following closed form solution for ϕ,

$$\phi = e^{-\delta T}\left(N(d_1) - \left(\frac{F_L e^{-gT}}{V(0)}\right)^{2\lambda} N(d_2)\right) + \frac{(1-\alpha_{BC})F_L e^{-gT}}{V(0)} I(T), \tag{A.13}$$

where $I(T)$ is given in equation (A.6), $\lambda = 1 + m/\sigma^2$ and

$$\begin{aligned} d_1 &= \frac{-\log\left(F_L e^{-gT}/V(0)\right) + \left(r-\delta-g+\sigma^2/2\right)T}{\sigma\sqrt{T}}, \\ d_2 &= \frac{\log\left(F_L e^{-gT}/V(0)\right) + \left(r-\delta-g+\sigma^2/2\right)T}{\sigma\sqrt{T}}. \end{aligned} \tag{A.14}$$

Similarly, the total bankruptcy costs in the dynamic model, $BC_L^{Dynamic}(0)$, are given by

$$BC_L^{Dynamic}(0) = \frac{BC_L(0)}{1-\phi}. \tag{A.15}$$

The total levered firm value, $TV_L^{Dynamic}(0)$, in the dynamic model equals the unlevered firm value $V\left(0\right)$, plus the total tax shields $TB_L^{Dynamic}$, less the total bankruptcy costs $BC_L^{Dynamic}(0)$,

$$TV_L^{Dynamic}(0) = V\left(0\right) + TB_L^{Dynamic}\left(0\right) - BC_L^{Dynamic}\left(0\right) = V\left(0\right) + \frac{TB_L\left(0\right) - BC_L\left(0\right)}{1-\phi}. \qquad \text{(A.16)}$$

The optimal capital structure is obtained by maximizing either the total firm value or the manager’s utility.