ESTIMATING A GENERAL STOCHASTIC VARIANCE MODEL FROM OPTION PRICES

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ABSTRACT

Methods have recently been devised for estimating from option prices a restricted class of stochastic variance models as well as deterministic variance models where the variance of the underlying asset is a deterministic function of the level of the underlying asset and time. Although the option prices generated by both of these types of models improve upon Black-Scholes values, their remaining empirical shortcomings motivate the development of a procedure for estimating more general models. This paper presents a method for estimating from option prices a general stochastic variance model. This model permits correlation between innovations to the level and the variance of the underlying asset as well as flexible nonparametric specifications of the market price of variance risk and the drift and diffusion functions of the variance process. A simulation experiment shows that the estimation procedure performs well on an artificial data set of the size available from the options markets. Estimation of the model from S&P500 index option prices over the period June 1, 1988 through December 29, 1995 provides evidence of misspecification in the restricted stochastic variance model.

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The Black-Scholes formula assumes that the dynamics of the underlying asset, S_t, obeys the following stochastic differential equation

$$\frac{dS_t}{S_t} = \mu(\bullet)dt + \sigma dW_t^S$$

where $\mu(\bullet)$ can be an arbitrary function of S_t and other economic variables[1], σ is a constant, and W_t^S is a standard Brownian motion. Despite the unquestionable importance and success of the Black-Scholes formula, it is known to have become increasingly unreliable over time (Rubinstein (1985) and Rubinstein (1994).) In response to this deteriorating performance, researchers have developed methods both to price options on underlying assets governed by richer dynamics and to estimate from option prices richer dynamics for the underlying asset.

Attempts to enrich the underlying dynamics have taken two main routes. The first of these relaxes the assumption that the diffusion coefficient is constant by allowing it to be a function of the current level of the underlying asset and time. Accordingly, this line of research takes the underlying asset to have dynamics described by

$$\frac{dS_t}{S_t} = \mu(\bullet)dt + \sigma(S_t, t)dW_t^S.$$

Dumas, Fleming, and Whaley (1998) refer to the resulting option pricing models as *deterministic volatility function (DVF)* models, because the instantaneous volatility is a deterministic function of the level of the underylying asset and time. This terminology will be adopted, although it

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[1] Merton (1973), Jagannathan (1984), Grundy (1991), and Lo and Wang (1995) all point out that that Black-Scholes formula does not depend on the drift being constant.

should be kept in mind that future instantaneous volatility is a random variable, because it is a function of the future level of the underlying asset which is a random variable.

Derman and Kani (1994), Dupire (1994), Rubinstein (1994), and Barle and Cakici (1995) develop methods to estimate $\sigma(S_t, t)$ nonparametrically by inferring a binomial or trinomial tree that exactly match a cross-section of observed option prices. Dumas, Fleming, and Whaley (1998) postulate a number of parametric forms for $\sigma(S_t, t)$ and estimate parameters by minimizing the sum of squared dollar errors between observed option prices and their *DVF* model values under the presumed parametric form. Empirically, the *DVF* models have fared poorly. Jackwerth and Rubinstein (1996b) study S&P500 index options and find that naïve (Black-Scholes based) volatility smile predictions outperform the *DVF* implied trees. Dumas, Fleming, and Whaley (1998) also investigate S&P500 index options and find that the predictive and hedging ability of their parametric *DVF* models does not exceed that of an *ad hoc* procedure that does nothing more than smooth the Black-Scholes implied volatilities across strike prices and times to expiration. They also find that simpler parametric forms are more successful than complicated forms which suggests that the latter suffer from overfitting – a problem that is likely to be exacerbated in the nonparametric implied trees.

Stochastic variance models are the second widely pursued approach to enriching the dynamics of the underlying asset from the Black-Scholes model. These models add a second source of randomness into the economy. A generic stochastic variance model can be written as follows

$$\frac{dS_t}{S_t} = \mu(S_t, V_t, t)\,dt + \sqrt{V_t}\,dW_t^S \tag{1}$$

$$dV_t = m(V_t)\,dt + \eta(V_t)\,dW_t^V \tag{2}$$

$$corr(dW_t^S, dW_t^V) = \rho \tag{3}$$

$$\lambda_{V,t} = \lambda_V(V_t) \tag{4}$$

$$r = constant \tag{5}$$

where dW_t^S and dW_t^V are standard Brownian motions and $\lambda_{V,t}$ is the market price of variance risk. $m(\bullet), \eta(\bullet)$, and $\lambda_V(\bullet)$ are arbitrary real-valued functions[2], and ρ is a real number between –1 and 1. Heston (1993) provides a closed-form (up to a single integration) formula for pricing European call options under model (1)-(5) when $m(\bullet), \eta(\bullet)$, and $\lambda_V(\bullet)$ are restricted to particular parametric forms. Willard (1997) develops a quick and accurate quasi-Monte Carlo method for valuing European call options under model (1)-(5) without placing any parametric restrictions on $m(\bullet), \eta(\bullet)$, and $\lambda_V(\bullet)$. Hence, given $m(\bullet), \eta(\bullet)$, and $\lambda_V(\bullet)$, and ρ, European call options can be priced under model (1)-(5).

Not nearly as much progress has been made on estimating model (1)-(5) from option prices. Renault and Touzi (1996) develop an iterative procedure for estimating the model under the restrictions that (1) $\rho = 0$, (2) $\lambda_{V,t} = 0$, and (3) $m(\bullet)$ and $\eta(\bullet)$ conform to particular parametric specifications (specifically, volatility follows an Ornstein-Uhlenbeck process). Pastorello, Renault, and Touzi (1997) estimate the model using an indirect inference method under the same set of restrictions. Despite their convenience from an econometric point of view, the three restrictions have no empirical or economic justification. Indeed, there is evidence that for many assets the correlation between innovations to the price level and innovations to the level

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[2] The range of $\eta(\bullet)$ is the positive real numbers. Different authors impose different regularity conditions on $m(\bullet), \eta(\bullet)$, and $\lambda_V(\bullet)$. For the time being, they are assumed to satisfy some set of regularity conditions which guarantee that the system of stochastic differential equations is well defined.

of instantaneous variance is substantial and negative.[3] Furthermore, there is certainly something odd about introducing stochastic variance into the model only to declare that investors do not care about it by stipulating *a priori* that the market price of variance risk is zero. Finally, although the mean-reversion inherent in the parametric forms chosen for $m(\bullet)$ and $\eta(\bullet)$ is empirically plausible, there is no reason other than tractability to choose one set of mean-reverting forms over another.

Beyond the problems with the restrictions when considered in isolation, there is empirical evidence that option pricing models which conform to them do not perform well. The large literature on volatility forecasting from options prices (see Figlewski (1997) for a review), imposes only the first two restrictions that $\rho = 0$ and $\lambda_{V,t} = 0$. This literature finds that across a large number of options markets the future volatility forecast from option prices is a significantly upwardly biased forecast of the actual realized future volatility.[4] Consequently, these papers reject the joint hypothesis of market efficiency and the stochastic variance model given in (1)-(5) under the restrictions $\rho = 0$ and $\lambda_{V,t} = 0$. The rejection is typically attributed either to the restriction on the correlation or the restriction on the market price of variance risk or both.

The empirical deficiencies of the *DVF* models and the restricted stochastic variance model in conjunction with the present ability to price options according to the (unrestricted) stochastic variance model (1)-(5) argue for the need for a method to estimate model (1)-(5). The task of this paper is to develop such a method. Section I develops the estimation method. Section II presents some simulations on artificial data that test the efficacy of the method.

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[3] This is the so-called leverage effect. See the literature begun by Black (1976).

[4] For tests of options on individual stocks see Lamoureux and Lastrapes (1993). For tests on index options see Fleming (1997). For tests of options on foreign currency futures see Jorion (1995).

Section III applies the method to S&P500 index option data. Section IV concludes. Some details of the estimation method are contained in the Appendix.

I. A Method for Estimating the Unrestricted Stochastic Variance Model

Previous papers that examine option market investor reaction to changes in instantaneous variance use the following model for the dynamics of an asset with price level S_t and instantaneous variance V_t that underlies a European call option $C(S_t,V_t,t)$:

$$dS_t = \mu(S_t, V_t, t)\ S_t\ dt + \sqrt{V_t}\ S_t\ dW_t^S$$

$$dV_t = m(V_t, t)\ dt + \eta(V_t, t)\ dW_t^V$$

$$corr(dW_t^S, dW_t^V) = 0$$

$$\lambda_{V,t} = 0$$

$$r = constant$$

where dW_t^S and dW_t^V are standard Weiner processes, and $\lambda_{V,t}$ is the market price of variance risk. Restricting the correlation of the Weiner processes and the market price of variance risk to zero is convenient, because it entails that the Black-Scholes implied volatility of an at-the-money (henceforth, ATM) call is very close to the square root of the average variance expected over the life of the option by a representative investor (see Feinstein (1988) for this result). Consequently, this model has the appealing feature that investor expectation of future variance can be extracted directly from ATM call prices without specifying or estimating the drift and diffusion functions of the variance process. This fact is put to use by Stein (1989) in the behavioral option pricing literature and by Fleming (1997), Jorion (1995), and Lamoureux and Lastrapes (1993) among others in the large literature on volatility forecasting.

Despite their convenience and widespread use, imposition of the restrictions $\rho = 0$ and $\lambda_{V,t} = 0$ has some serious drawbacks. First, they have no empirical justification. Indeed, there is evidence that for many assets the correlation between innovations to the price level and innovations to the level of instantaneous variance is substantial and negative.[5] Second, they have no economic justification. There is certainly something odd about introducing stochastic variance into the option pricing model only to declare *a priori* that investors do not care about it by stipulating that $\lambda_{V,t} = 0$. Third, they make it extremely difficult to interpret any rejection of the joint hypothesis of market efficiency and the stochastic variance model that they define. For example, in his discussion of the bias he finds in option-based forecasts of future variance Fleming (1997) writes of the zero correlation constraint that "although [the] bias may suggest option market inefficiency, it may also stem from misspecification … in the option valuation model." Lamoureux and Lastrapes (1993), on the other hand, believe that the similar bias that they find may be attributable to their assumption that the market price of variance risk is zero. They suggest that

> Given informational efficiency, our results can be explained by the existence of a risk premium applied to a nontraded variance process … Therefore, further attempts to learn from the data should explicitly model a risk premium on the variance process.

Finally, even the potential advantage of the restrictions – that they make it possible to extract information about investor expectations of future variance without specifying and estimating the

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[5] This is the so-called leverage effect. See the literature spawned by Black (1976).

drift and diffusion functions of the variance process – is not useful in the present context. The reason is that the hypothesis to be examined in this paper involves investor reaction to changes in instantaneous variance, and instantaneous variances cannot be deduced from option prices via the above model in the absence of estimates of the drift and diffusion functions.[6]

In order to avoid the pitfalls identified above, the present paper will assume a model that allows innovations to the level and instantaneous variance of the underlying asset to be correlated and the market price of variance risk to be non-zero. In particular, the following model of market equilibrium will be assumed throughout this paper:

$$dS_t = \mu(S_t, V_t, t)\, S_t\, dt + \sqrt{V_t}\, S_t\, dW_t^S \tag{1}$$

$$dV_t = m(V_t)\, dt + \eta(V_t)\, dW_t^V \tag{2}$$

$$corr(dW_t^S, dW_t^V) = \rho \tag{3}$$

$$\lambda_{V,t} = \lambda_V(V_t) \tag{4}$$

$$r = constant. \tag{5}$$

As before dW_t^S and dW_t^V are standard Weiner processes, and $\lambda_{V,t}$ is the market price of variance risk. ρ is constant, and there are no imperfections in the market (i.e., securities are infinitely divisible; there are no taxes; trading is continuous; there are no bid-ask spreads, price discreteness, or transaction costs; etc.). The only constraint that has been added to the previous model is that the variance process has been specified to be time homogeneous. This mild condition is necessary in order to get repeated observations for estimation purposes.

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[6] Pastorello, Renault, and Touzi (1997) provide evidence that Black-Scholes implied variances are not good proxies for current instantaneous variance. Furthermore, a time-series of instantaneous variances (rather than Black-Scholes implied variance proxies) will also be needed in order to test whether the variance process is Markovian which will be important for interpreting the test results reported below.

Since the underlying asset S is assumed to be traded, there is no need to estimate $\mu(S_t, V_t, t)$. The strategy that will be pursued to estimate $m(V)$, $\eta(V)$, $\lambda_V(V)$, and ρ will make use of two facts. First, if $m(V)$, $\eta(V)$, $\lambda_V(V)$, and ρ are known, then under model (1)-(5) observing an option price is equivalent to observing the current level of instantaneous variance. Second, if a time-series of instantaneous variances is known, then $m(V)$, $\eta(V)$, $\lambda_V(V)$, and ρ can be estimated directly from the time-series.[7] Of course, initially neither $m(V)$, $\eta(V)$, $\lambda_V(V)$, and ρ nor the time-series of instantaneous variances is known. Consequently, the functions $m(V)$, $\eta(V)$, and $\lambda_V(V)$, the value of ρ, and the instantaneous variance on each trade date will be estimated simultaneously via an iterative EM-type algorithm.

The initial step of the algorithm is an expectation or *E* step that estimates a daily time-series of instantaneous variances by assigning to each trade date the average of Black-Scholes implied variances of liquid call options observed on that trade date. All subsequent *E* steps use the $m(V)$, $\eta(V)$, and $\lambda_V(V)$ functions and ρ value estimated in the previous step to assign as the instantaneous variance for each trade date the average of the instantaneous variances implied on that trade date from liquid call options under model (1)-(5). The maximization or *M* step uses the time-series of instantaneous variances from the previous *E* step to estimate the functions $m(V)$ and $\eta(V)$ and the value of ρ. The *M* step also estimates $\lambda_V(V)$ by using on each trade date the prices of two close to ATM calls that are identical except for a one month difference in time to maturity.

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[7] Estimating $\lambda_V(V)$ requires also observing two option prices on each trade date. Estimating ρ requires that a time-series of the level of the underlying asset is also known.

Let Δ be the time interval between trade dates, and assume that observations are available on the $T+1$ consecutive trade dates $\{0\Delta, 1\Delta, 2\Delta, ..., T\Delta\}$. The EM-type algorithm then consists of repeated application of the following two steps:

> **ith step (*E* step)**: Use $m(V)$, $\eta(V)$, $\lambda_V(V)$, and ρ from step i-1 to infer a daily time-series of instantaneous variances $\{V_{0\Delta}, V_{1\Delta}, V_{2\Delta}, ..., V_{T\Delta}\}$ from liquid options on each trade date under the assumption that model (1)-(5) is valid. (The first step of the algorithm is an *E* step which estimates $\{V_{0\Delta}, V_{1\Delta}, V_{2\Delta}, ..., V_{T\Delta}\}$ by the Black-Scholes implied variance of liquid option on each trade date.)
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> **i+1st step (*M* step)**: Using $\{V_{0\Delta}, V_{1\Delta}, V_{2\Delta}, ..., V_{T\Delta}\}$ from step i, estimate $m(V)$, $\eta(V)$, and ρ. Then given these estimates of $m(V)$, $\eta(V)$, and ρ, estimate $\lambda_V(V)$ from the prices on each trade date of two close to ATM calls with a one month difference in time to maturity.

The *E* step produces an instantaneous variance on each trade date by averaging the instantaneous variances implied by the prices of liquid options given $m(V)$, $\eta(V)$, $\lambda_V(V)$, and ρ. Willard (1997) develops a quasi-Monte Carlo method (i.e. a method of numerical integration) that produces accurate call prices under model (1)-(5) on a standard PC in less than one second. Willard's method accomplishes quick and accurate pricing by adding four variance reduction techniques to standard Monte Carlo option pricing: conditioning on variance paths, using low-discrepancy sequences, employing the so-called *Ciesielski* discretization, and using antithetic variables. These variance reduction techniques are discussed in subsection A of the Appendix. Bajeux and Rochet (1992) prove that under the presumed model call prices are strictly monotonically increasing in the current level of instantaneous variance.[8] As a result, a standard numerical minimization routine can be used in conjunction with Willard's quasi-Monte Carlo pricing method to infer a time-series of instantaneous variances from call prices.

The *M* step estimates $m(V)$, $\eta(V)$, $\lambda_V(V)$, and ρ given the daily time series of instantaneous variances from the previous *E* step, $\{V_{0\Delta}, V_{1\Delta}, V_{2\Delta}, \ldots, V_{T\Delta}\}$. ρ is estimated by the sample correlation between $\{V_{0\Delta}, V_{1\Delta}, V_{2\Delta}, \ldots, V_{T\Delta}\}$, and the daily time series of S&P500 index level values, $\{S_{0\Delta}, S_{1\Delta}, S_{2\Delta}, \ldots, S_{T\Delta}\}$. $m(V)$ and $\eta(V)$ are estimated nonparametrically from the daily time-series of instantaneous variances using the method of Stanton (1997). The central fact used by Stanton's method is that when the variance process is represented by equation (2) then under suitable restrictions on $m(V)$ and $\eta(V)$ the conditional expectation of an arbitrary function $f(V,t)$ has a Taylor's series expansion

$$E_t\left[f(V_{t+\Delta}, t+\Delta)\right] = f(V_t, t) + Lf(V_t, t)\Delta + \frac{1}{2}L^2 f(V_t, t)\Delta^2 + \ldots + \frac{1}{n!}L^n f(V_t, t)\Delta^n + O(\Delta^{n+1}) \quad (6)$$

where L is the infinitesmal generator and Lf is the expected infinitesmal rate of change of the function f and is given by the formula

$$Lf(V,t) = \frac{\partial f(V,t)}{\partial t} + \frac{\partial f(V,t)}{\partial V} m(V) + \frac{1}{2}\frac{\partial^2 f(V,t)}{\partial V^2}\eta^2(V) \quad (7)$$

When $m(V)$ and $\eta(V)$ are assumed to be known equations (6) and (7) are often used to construct numerical approximations of the expectation $E_t\left[f(V_{t+\Delta}, t+\Delta)\right]$. Here these equations will be used in the opposite way. The expectations on the left hand side of (6) will be estimated nonparametrically from the data and the choice of a suitable set of functions *f* will make it possible to invert out the drift and diffusion functions $m(V)$ and $\eta(V)$. The details of the choice of functions and the subsequent inversion are provided in the Appendix. The resulting third order formulas which will be used to estimate the drift and diffusion functions are

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[8] They actually prove monotonicity only for the case of $\rho = 0$, but the result should extend easily to the case of

$$m(V)=\frac{1}{6\Delta}\left\{18E\left[V_{t+\Delta}-V_t\middle|V_t=V\right]-9E\left[V_{t+2\Delta}-V_t\middle|V_t=V\right]+2E\left[V_{t+3\Delta}-V_t\middle|V_t=V\right]\right\}+O\left(\Delta^3\right) \tag{8}$$

and

$$\eta(V)=\sqrt{\frac{1}{6\Delta}\left\{18E\left[\left(V_{t+\Delta}-V_t\right)^2\middle|V_t=V\right]-9E\left[\left(V_{t+2\Delta}-V_t\right)^2\middle|V_t=V\right]+2E\left[\left(V_{t+3\Delta}-V_t\right)^2\middle|V_t=V\right]\right\}}+O\left(\Delta^3\right) \tag{9}$$

where the expectations are computed nonparametrically using kernel estimation. For example, the first expectation in the expression for *m(V)* is computed by:

$$E[V_{t+\Delta}-V_t\mid V_t=V]\approx\frac{\sum_{t=0}^{T-1}\left(V_{(t+1)\Delta}-V_{t\Delta}\right)K\left[\left(V-V_{t\Delta}\right)/h\right]}{\sum_{t=0}^{T-1}K\left[\left(V-V_{t\Delta}\right)/h\right]}$$

where the kernel $K[\bullet]$ is a standard normal density function and *h* is a bandwidth that is chosen to minimize the asymptotic mean integrated square error of the estimated density function. See Scott (1992) or Silverman (1986) for detailed expositions of kernel estimation.

An approach similar to that taken in Stanton (1997) to find the market price of interest rate risk will be used to estimate the market price of variance risk, $\lambda_V(V)$. The market price of variance risk, $\lambda_V(V)$, can be regarded as specifying the excess return required by investors to bear an extra unit of variance risk when the level of the variance is *V*. Consequently, the difference in excess returns of two options that are identical except for a one month difference in time to maturity is informative about the market price of variance risk. A procedure similar to the one that is used to estimate *m(V)* and $\eta(V)$ is developed in the Appendix to estimate $\lambda_V(V)$ based on daily data on the excess returns of close to ATM options that are identical except for a one month difference in time to maturity. The final expression for estimating $\lambda_V(V)$ is given by

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ρ = constant.

$$\lambda_V = \frac{1}{\eta(V)\left[\frac{C_V^A(S,V,t)}{C^A(S,V,t)} - \frac{C_V^B(S,V,t)}{C^B(S,V,t)}\right]}\left\{\frac{1}{6\Delta}\left[18E_t\left[R_{t,t+\Delta}^{AB}(S,V,t)\right] - 9E_t\left[R_{t,t+2\Delta}^{AB}(S,V,t)\right] + 2E_t\left[R_{t,t+3\Delta}^{AB}(S,V,t)\right]\right] - \left[\mu(S,V,t) - r\right]\left[\frac{SC_S^A(S,V,t)}{C^A(S,V,t)} - \frac{SC_S^B(S,V,t)}{C^B(S,V,t)}\right]\right\} + O(\Delta^3) \tag{10}$$

where

$$R_{t,t+\tau}^{AB}(S,V,t) \equiv \text{The difference in the returns on calls } A \text{ and } B \text{ between } t \text{ and } t+\tau$$

and on a given trade date options *A* and *B* are short maturity calls that are closest to ATM and have identical characteristics except for a one month difference in time to maturity. As in the case of *m(V)* and $\eta(V)$, the various components of the expression for λ_V are estimated nonparametrically via kernel estimation. Since it is difficult to estimate derivatives nonparametrically the C_S/C and C_V/C ratios that appear in equation (10) are replaced by expressions involving covariances between option returns and either changes in *S* or *V*. The exact expressions that are substituted are derived in the Appendix.

The proposed EM-type algorithm is a generalization of one that has already been studied. Renault and Touzi (1996) consider a model where $\rho = 0, \lambda_{V,t} = 0$, and the volatility follows an Ornstein-Uhlenbeck process (i.e. volatility has a linearly mean reverting drift function and a constant diffusion function). They prove that when an EM-type algorithm analogous to the one described here is used to estimate the parameters in the drift and diffusion functions, the algorithm is a strong contraction in the neighborhood of the true values of the parameters. Pastorello, Renault, and Touzi (1994) implement the EM-type algorithm on simulated options data.

II. Data and Estimation Results

S&P500 Index options are studied, because they constitute the most liquid European options market. These options trade under the symbol *SPX* with expiration dates in the three near term months along with the following three months from the March expiration cycle (March, June, September, December). The actual expiration day is the Saturday immediately following the third Friday of the month. Strike price intervals are 5 points for near months and 25 points for far months. The minimum tick for options trading below $3.00 is 1/16 and 1/8 for options trading at higher prices.

Daily data on S&P500 Index options were obtained from the Center for Research in Security Prices (CRSP) at the University of Chicago. The CRSP data comes from the Chicago Board Options Exchange (CBOE) where the options are traded. The data used for each trade date is the closing price, exercise price, time-to-expiration, and volume for each option trading on that day. The initial sample considered is the 159,469 S&P500 Index options with positive daily trading volume for the period June 1, 1988 through December 29, 1995. Data is available beginning October 2, 1985. The data from October 2, 1985 through May 31, 1988 is not used because of the evidence presented in Jackwerth and Rubinstein (1996a) that there is a structural break in the SPX options market data at the time of the October 1987 stock market crash. In addition, the market was considerably less liquid during its earlier years, and as a result the data may not be as reliable.

The risk-free rate of interest associated with each option is derived from the CRSP Daily U.S. Government Bond File. For each option, the bid and ask prices for the treasury bill which trades on the same trade date as the option and whose time to maturity is closest to the time to expiration of the option are determined. If the time to maturity of the treasury bill is less than ten calendar days, then the bid and ask price of the shortest maturity treasury bill with maturity greater than nine calendar days is used. The bid and ask prices are converted into interest rates and their average is taken to be the risk-free rate for the option.

SPX options unlike OEX (S&P100) options are European and do not have a wildcard feature. These facts considerably simplify empirical work with SPX options. Nonetheless, two serious challenges remain. The first challenge is matching observations on option prices to observations on the underlying index level. Even if a quote on the underlying index can be exactly matched temporally to an option price, the quoted index level will not be the proper underlying value for the option since all 500 underlying prices will not correspond to trades that occurred at the quoted time. One way to try to circumvent this problem would be to use prices on S&P500 index futures which trade on the Chicago Mercantile Exchange (CME). This approach, however, is not promising. Since the CBOE and CME close at different times, using closing prices on the futures and options is not a viable strategy. Even if the recorded times of time-stamped quotes can be matched across the two markets, there is no guarantee that they will be perfectly synchronized and even small differences in timing can produce non-trivial changes in the implied instantaneous variance or moneyness computed for an option.

The second challenge is determining the *expected* future rate of dividend payments by the stocks that compose the index until the expiration of an option. Of course, dividend rates can always be calculated after the fact from the actual dividends paid out by the S&P500 stocks. The ex-post rate, however, may not match the ex-ante expectation at the time the option is priced.

These two challenges will be met by adapting a method employed by Aït-Sahalia and Lo (1998) that utilizes the spot-futures parity and put-call parity relationships. To understand the method let D be the present value of dividends expected to be paid out by the underlying index until the expiration of an option and let δ be the rate at which the underlying index is expected to pay dividends. Then familiar formulas for spot-futures and put-call parity are:[9]

$$F = (S - D)e^{rT} \tag{11}$$

$$F = Se^{(r-\delta)T} \tag{12}$$

$$c + D + Ke^{-rT} = p + S \tag{13}$$

Combining (11) and (13) yields:

$$c + Ke^{-rT} = p + Fe^{-rT}$$

or

$$F = (c - p)e^{rT} + K \tag{14}$$

Substituting for F from (12) gives:

$$Se^{-\delta T} = c - p + Ke^{-rT} \tag{15}$$

Equations (14) and (15), respectively, permit the computation of the moneyness or the price of options according to some model without making use of the level of or the dividends paid by the underlying index. Since the call and the put are traded in the same market, there is no synchronization problem from using closing prices for c and p as long as liquid options are employed.

The moneyness of an option, m, is defined as its strike price divided by the futures prices. As a result, equation (14) implies that the moneyness can be computed by

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[9] See Hull (1993), p. 53 for (11), Hull (1993), p. 55 for (12), and Hull (1993), p. 167 for (13).

$$m \equiv \frac{K}{F} = \frac{K}{(c-p)e^{rT} + K}. \tag{16}$$

The price of a European option written on an asset with price *S* and dividend yield δ is the same as the price of a European option written on an asset with a price of $Se^{-\delta T}$ that pays no dividend.[10] As a result, the method that will be used throughout this paper to compute option prices under model (1)-(5) will be to replace *S* with the value of $Se^{-\delta T}$ calculated from (15) and then to proceed to price options as if the underlying index pays no dividends.

The *E* step of the EM-type algorithm requires that on each trade date the instantaneous variance for that trade date is implied from the price of liquid options under model (1)-(5). In principle, the instantaneous variance could be implied from a single option on each trade date. The daily data from the CBOE, however, supplies only the last transaction price for each option. There is no way to know whether that price is at the bid, the ask, or somewhere else. As a result, bid-ask bounce as well as price discreteness will introduce spurious variance into the time-series of implied instantaneous variance if only one option price is used on each trade date.[11] In order to mitigate this problem, the *E* step assigns as the instantaneous variance for each trade date the average of the instantaneous variances implied from four put-call parity pairs under model (1)-(5). The four pairs used are the two which are closest to ATM (i.e. to *m*=1) among those with shortest maturity greater than nine calendar days and the two which are closest to ATM among those with then next greater maturity. Equation (14) is used to avoid employing data on the level of the underlying index or the dividends that are expected to be paid by the S&P500 index until

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[10] See Hull (1993), p. 248. The reason is that a stock that grows from *S* at time *0* to S_T at time *T* will grow from $Se^{-\delta T}$ at time *0* to S_T at time *T* if it retains all of the dividends as capital gains.

[11] Figlewski (1997) provides examples which show that the impact of the bid-ask spread and price discreteness on implied variances may be large.

expiration of the option when computing the moneyness of candidate put-call parity pairs. Once the four put-call parity pairs are chosen, the value of $Se^{-\delta T}$ is computed for each pair from equation (15) and it is then assumed that $Se^{-\delta T}$ is the level of the underlying index and that the index does not pay dividends. Under these assumptions and the estimation of model (1)-(5) from the previous *M* step, instantaneous variances are implied from each of the call prices using a standard numerical search algorithm in conjunction with the quasi-Monte Carlo technique discussed in Section I and the Appendix. The instantaneous variance assigned to each trade date is the average of the four instantaneous variances implied from the call prices.[12]

The main data for the *M* step is the daily time-series of implied instantananeous variances from the previous *E* step. The drift and diffusion functions, $m(V)$ and $\eta(V)$, are estimated directly from this time-series via equations (8) and (9). A daily time-series of the level of the S&P 500 index is needed in order to compute ρ, the correlation between innovations to the instantaneous variance and the level of the index. Daily data on the S&P500 index is obtained from the CRSP Indices File. Note that the only place that data on the level of the S&P500 index is used is in computing this correlation. Computing $\lambda_V(V)$ from equation (10) requires two put-call parity pairs on each trade date that are idential except for a one month difference in expiration. The put-call parity pairs used are the closest to ATM with shortest time to expiration greater than nine calendar days and less than 46 calendar days and the pair with the same strike price that expires the next month. If both of these pairs are not available, then the second closest to ATM pair is used with shortest time to expiration greater than nine calendar days and less than 46 calendar

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[12] The initial *E* step assigns to each trade date the average of the Black-Scholes implied variances of the four calls from the specified put-call parity pairs. When implying these variances it is asssumed that the level of the underlying index is $Se^{-\delta T}$ and that the index pays no dividends.

days and the pair that expires the next month. If either of these pairs is unavailable, then the trade date is not used in the estimation of $\lambda_V(V)$. These put-call parity pairs are used to estimate $\lambda_V(V)$ according to equation (10) as described in Section I and the Appendix.

The EM-type algorithm converged quickly. Fourteen iterations were performed, although convergence was essentially achieved by the fourth iteration. The results of the final iteration will be presented first, and then the convergence will be examined.

Figure 1 shows the time-series of daily implied instantaneous variances from the first and fourteenth iterations. The qualtitative features of the two time-series are similar. It can be seen, however, that trade dates that initially had lower values had their values reduced and trade dates that initially had higher values had their values increased. Figure 2 examines this phenomenon more closely by providing plots of the time-series of daily implied instantaneous variance for the fourteenth iteration and of the difference of the fourteenth and first iteration time-series. The observed effect can be explained by recalling that instantaneous variance is mean-reverting and that the Black-Scholes implied variances which are used as proxies for the instantaneous variance in the first iteration are roughly the average variances expected over the life of the options.[13] As a result, when the instantaneous variance on a trade date is high, the average variance expected over the life of the option is somewhat lower. Likewise, when the instantaneous variance on a trade date is low, the average variance expected over the life of the option is somewhat higher. Accordingly, the time-series of actual instantaneous variances (found in the 14th iteration) in effect expands the time-series of Black-Scholes implied variances (found in the 1st iteration).

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[13] The Black-Scholes implied variance is almost exactly equal to the average variance expected over the life of an ATM call in a model where ρ=0 and the market price of variance risk is zero. Under model (1)-(5) the relationship is less exact.

Figures 3, 4, and 5 provide, respectively, third-order approximations to the drift, diffusion, and market price of variance risk functions. The ninety-five percent confidence intervals are computed using 10,000 iteration of Künsch's (1989) block boostrap algorithm. The correlation between innovations to the level and innovations to the instantaneous variance of the S&P500 index was –0.63 for the final iteration.

Figure 3 reveals that the drift function, $m(V)$, of the variance process implicit in the SPX options is not linear. The rightward pull of the drift function is approximately constant when the instantaneous variance is in the range zero to 0.025.[14] The drift function begins declining sharply when the variance is greater than 0.025 and it becomes negative at about 0.033. As will be seen, the risk-neutral drift function is also non-linear. This finding may have important implications for standard stochastic variance option pricing models such as Heston (1993), Bates (1996), and Scott (1997) that specify a linear drift function for the variance process.

Figure 4 shows that the diffusion function, $\eta(V)$, is increasing and convex. This again is in conflict with the square-root specification of the diffusion function of the variance process used in the standard stochastic variance option pricing models. The impact of bid-ask spread and price discreteness on the estimated diffusion function is unknown. Future work will investigate this issue by using transaction rather than daily data on the SPX index options.

Figure 5 plots the market price of variance risk, $\lambda_V(V)$, and $-\lambda_V(V)\eta(V)$ which is the quantity that must be added to the drift function $m(V)$ to obtain the risk-neutral drift function. The market price of variance risk is negative throughout most of its range which indicates that there is a positive premium for bearing variance risk. The market price of variance risk decreases for small value of the instantaneous variance and increases for large value of instantaneous variance. Hence, it is nonlinear and like the drift and the diffusion functions does not conform to the specification in the standard stochastic variance option pricing model. The plot of the quantity $-\lambda_V(V)\eta(V)$ is increasing for small values of instantaneous variance and increasing for large values of instantaneous variance. Consequently, the nonlinearity which was observed in the estimated drift function will be exacerbated when $-\lambda_V(V)\eta(V)$ is added to it to obtain the risk-neutral drift function.

The EM-type algorithm converges quickly. Figure 6 displays the risk-neutral drift functions and the diffusion functions from all fourteen iterations of the algorithm. It can be seen for both the risk-netural drift and the diffusion functions the largest jump is between the first and the second iteration and that convergence has been essentially achieved by the fourth iteration which is not easily distinguishable from the higher iterations. The impression that the algorithm has basically converged by the fourth iteration is confirmed by Figure 7 which shows only the fourth and the fourteenth iteration of the risk-neutral drift and the diffusion functions. For both of these functions the fourth and the fourteenth iterations can barely be distinguished. Figure 8 shows that the thirteenth and the fourteenth iterations cannot be distinguished. Table I reports descriptive statistics on the difference in the instantaneous variance implied on consecutive iterations of the algorithm for each of the 1917 trade dates from June 1, 1988 through December 29, 1995. The third column gives the mean of the absolute difference for the 1917 trade dates. It can be seen that the mean absolute difference is by far the largest between the second and the first iteration. By the time four iterations have been completed, the mean absolute difference is

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[14] Although when determining the total rightward pull on a process at some level both the drift and the diffusion at

less than 0.0001. In other words, after the fourth iteration the average change in the instantaneous variance on each trade date is less than 0.0001. Figure 9 shows the progression of the correlation between innovations to the level and innovations to the instantaneous variance of the S&P500 index. The correlation falls between –0.68 and –0.63 for all fourteen iterations. The correlation remains very close to –0.63 after the fourth iteration.

Appendix

This Appendix is comprised of three subsections. Subsection A outlines the variance reduction techniques used by Willard's (1997) quasi-Monte Carlo option pricing method. Subsection B derives Stanton's (1997) third order nonparametric estimators for the drift and diffusion functions of a diffusion process. Subsection C explicates the procedure used to obtain a nonparametric estimate of the market price of variance risk.

A. Variance Reduction in Willard's (1997) quasi-Monte Carlo Option Pricing Method

A.1 Conditioning

Under the risk-neutral measure the stock and variance process in model (1)-(5) can be represented as:

$$dS_t^* = rS_t^* dt + \sqrt{V_t}\, S_t^* \, dW_t^{S*} \tag{A1}$$

$$dV_t^* = \left[m(V_t^*) - \lambda_V({}_t V_t^*)\eta(V_t^*)\right]dt + \eta(V_t^*)\,dW_t^{V*} \tag{A2}$$

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that level need to be taken into account. See Conley, Hansen, Luttmer, and Scheinkman (1997).

$$corr(dW_t^{S*}, dW_t^{V*}) = \rho \, . \qquad \text{(A3)}$$

Under these dynamics, the price of the stock at time *T* given its price at time *0* is:

$$S_T^* = S_0^* \xi \exp\left[rT - \frac{\left(1-\rho^2\right)}{2} \int_0^T V_u^* du + \sqrt{1-\rho^2} \int_0^T \sqrt{V_u^*} dW_u^{S*} \right]$$

where the random variable ξ is defined by:

$$\xi \equiv \exp\left[-\frac{\rho^2}{2} \int_0^T V_u^* du + \rho \int_0^T \sqrt{V_u^*} dW_u^{V*} \right]$$

Consequently, conditional on a variance path between times *0* and *T*, W^V, ξ is a constant and $logS_T$ is normally distributed with mean

$$\log S_0^* \xi + rT - \frac{\left(1-\rho^2\right)}{2} \int_0^T V_u^* du$$

and variance

$$\left(1-\rho^2\right) \int_0^T V_u du.$$

As a result, conditional on the path of the Weiner process W^V the price of a European call option is given by the Black-Scholes formula with $S_0 \xi$ plugged in for the current stock price and $\sqrt{\frac{\left(1-\rho^2\right)}{T} \int_0^T V_u du}$ plugged in for the volatility. Using the law of iterated expectations then gives the unconditional price of the option as

$$E_{W^V}\left[C^{BS}\left(S_0^* \xi, K, r, T, \sqrt{\frac{\left(1-\rho^2\right)}{T} \int_0^T V_u du} \right) \right]$$

where the expectation is taken with respect to paths of the variance process. This final formula allows the price of European call options to be determined under the model (1)-(5) by simulating only the variance process provided that $m(V)$, $\eta(V)$, $\lambda_V(V)$, and ρ are known. The price is determined by averaging the Black-Scholes prices that result from the stock price and volatility adjustments entailed by each simulated variance path.

A.2 Low-Discrepancy Sequences

Under standard Monte Carlo the variance paths required to compute the option prices via the conditioning described in the previous subsection are constructed by converting standard normal deviates produced by a pseudo-random number generator into increments of the variance process ΔW^{V*}. This procedure is inefficient, because the draws from the pseudo-random number generator tend to cluster which results in sample paths that clump together relative to their true distribution. Quasi-Monte Carlo methods are similar to standard Monte Carlo methods except that instead of drawing normal deviates at random, they use a deterministic set of point in order to fill in the range of possble paths in a way that roughly mirrors their true distribution.

In order to get an idea of the mechanics of a quasi-Monte Carlo simulation suppose that 500 variance paths are desired and each one is constructed by dividing the interval between 0 and T into 64 pieces. The first step is to produce 64 sequences of 500 numbers in the interval [0,1]. These sequences should be generated so that (1) the 500 numbers within each sequence are as far apart from each other as possible, and (2) the sequences are as different from one another as possible. The goal is both to fill in the interval [0,1] associated with each sequence as uniformly as possible and at the same time to fill in the unit hypercube associated with all of the sequences as uniformly as possible. The term *discrepancy* refers to some measure that is increasing in the degree of clustering of the points in the hypercube. There are a number of methods for constructing low-discrepancy sequences. The Halton sequence was used in this paper, and the reader interested in the details of this or other sequences is referred to Galanti and Jung (1997) and the references cited there.

Once the 64 sequences are generated each of the 64*500 numbers on the unit interval is transformed into the analog of a standard normal deviate by applying the inverse normal function.[15] These standard normal deviate analogs are then substituted into the simulation procedure in place of the pseudo-random standard normal draws.

A.3 The Ciesielski Discretization

For all known methods of generating low-discrepancy sequences, the quality of the sequences deteriorate as the number of sequences generated (which corresponds here to the number of steps into which the time interval *0* to *T* is broken) increases. The effect of this deteriorization is mitigated by using the Ciesielski discretization to use the highest quality sequences at the economically most significant points of the Weiner process. The Ciesielski discretization accomplishes this by seeing to it that the highest quality samples of the variance process are obtained at the points where its variance are greatest.

Since the variance of a Weiner process is linear in the time elapsed since it has been observed, the first sequence generated is used to simulate the Weiner process at time *T*. In particular, set $W_T^{V*} = \sqrt{T}\,N_1$ where N_1 is the value of a point of the first low-discrepancy sequence after the inverse normal function has been applied to it. This pins down the Weiner process at time *0* and at time *T*. Hence, it is a Brownian Bridge process which has its greatest

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[15] It is crucial to have a very accurate computation of the inverse normal function. The inverse normal functions built into Matlab and Mathematica are not sufficient. The second algorithm described in Wichura (1988) which is accurate to about sixteen decimal places over the range min[p, 1-p]>10^{-316} was used for the analysis in this paper.

variance at time *T/2*. Accordingly, the second sequence is used to simulate the Weiner process at time *T/2*. Once W_0^{V*} and W_T^{V*} are fixed the process at time *T/2* is normally distributed with mean $W_T^{V*}/2$ and variance *T/4*. As a result the Weiner process is simulated at time *T/2* by

$$W_{T/2}^{V*} = \frac{W_T^{V*}}{2} + \sqrt{\frac{T}{4}} N_2$$

where N_2 is the value of a point of the second low-discrepancy sequence after the inverse normal function has been applied to it. As soon as the Weiner process is simulated at time *T/2* there are two Brownian Bridge processes, one from time *0* to time *T/2* and the other from time *T/2* to time *T*. These Brownian Bridge processes have their greatest variance at their midpoints at times *T/4* and *3T/4*. The third and fourth sequences are used to simulate the Weiner processes at these points as above. By continuing to follow this procedure there is an exact match between the economic importance of points on the Weiner process and the quality of the sequences used to simulate them. Caflisch and Moskowitz (1994) are the first to use the Ciesielski discretization in conjunction with low-discrepancy sequences.

A.4 Antithetic Variables

The use of antithetic variables is a widely used Monte Carlo variance reduction technique. In the present context this technique doubles the number of variance paths that are used to calculate the option price by using both the original set of low-discrepancy sequences and the set that is obtained by replacing each number in the original low-discrepancy sequences by one minus that number. See Boyle (1977) for an exposition of the antithetic variable technique.

B. Nonparametric estimation of m(V) and η(V).

This subsection of the Appendix explains how to invert equations (6) and (7) to obtain the drift and diffusion functions *m(V)* and *η(V)*. The method was first developed in Stanton (1997) but the exposition here follows Boudoukh, Richardson, Stanton, and Whitelaw (1998).

First define the quantity

$$\hat{E}^{i}(V_t,t) \equiv \frac{1}{i\Delta} E_t\left[f(V_{t+i\Delta},t+i\Delta)-f(V_t,t)\right] \quad \textit{for } i=1,2,\ldots$$

Then re-arranging equation (6) gives

$$\hat{E}^{i}(V_t,t) = \mathcal{L}f(V_t,t)+\frac{1}{2}\mathcal{L}^2 f(V_t,t)(i\Delta)+\ldots+\frac{1}{n!}\mathcal{L}^n f(V_t,t)(i\Delta)^{n-1}+O(\Delta^n)$$

This shows that each of the $\hat{E}^{i}(V_t,t)$ is a first order approximation to $\mathcal{L}f(V_t,t)$. The task now is to form linear combinations of the $\hat{E}^{i}(V_t,t)$ to produce order *N* approximations to $\mathcal{L}f(V_t,t)$. Consider the following linear combination of the previous equation

$$\sum_{i=1}^{N}\alpha_i\hat{E}^{i}(V_t,t) = \left[\sum_{i=1}^{N}\alpha_i\right]\mathcal{L}f(V_t,t)+\frac{1}{2}\left[\sum_{i=1}^{N}\alpha_i i\right]\mathcal{L}^2 f(V_t,t)\Delta+\frac{1}{6}\left[\sum_{i=1}^{N}\alpha_i i^2\right]\mathcal{L}^3 f(V_t,t)\Delta^2+\ldots \qquad \text{(A.4)}$$

In order to get the order *N* approximation, the length *N* vector α must satisfy the following matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & N \\ 1 & 4 & 9 & \cdots & N^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{N-1} & 3^{N-1} & \cdots & N^{N-1} \end{pmatrix}\alpha \equiv Z\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Z is a Vandermonde matrix which is invertible for any value of *N*. Hence, α is obtained via

$$\alpha = Z^{-1}\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}$$

For *N=3*, the solution to this equation is:

$$\alpha = \begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 4 & 9\end{pmatrix}^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\-3\\1\end{pmatrix}$$

Substituting this into (A.4) and using the definition of $\hat{E}^i(V_t, t)$ gives

$$\mathcal{L}f(V_t,t) = \frac{1}{6\Delta}\{18E_t[f(V_{t+\Delta}, t+\Delta) - f(V_t,t)] - 9E_t[f(V_{t+2\Delta}, t+2\Delta) - f(V_t,t)] + 2E_t[f(V_{t+3\Delta}, t+3\Delta) - f(V_t,t)]\} + O(\Delta^3) \tag{A.5}$$

To approximate a particular function, *g*(*V*), all that it needed is a function *f* that satisfies

$$\mathcal{L}f(V_t,t) = g(V).$$

Using the expression given for the infinitesmal generator in equation (7), it can be seen that the functions

$$f_1(V,t) = V$$
$$f_2(V,t) = (V - V_t)^2$$

have infinitesmal generators

$$\mathcal{L}f_1(V,t) = m(V)$$
$$\mathcal{L}f_2(V,t) = \eta^2(V).$$

Plugging these equations into (A.5) yield equations (8) and (9) in the text.

C. Nonparametric Estimation of the Market Price of Variance Risk

This subsection of the Appendix develops an expression for nonparametric estimation of the market price of variance risk.

Let *C(S,V,t)* be the price of an asset that depends on *S, V,* and *t.* Then by a standard argument from the option pricing literature, model (1)-(5) implies:

$$C_t + r\ S\ C_S + C_V\left[m(V) - \lambda_V\ \eta(V)\right] + \frac{1}{2} V\ S^2 C_{SS} + \frac{1}{2}\eta(V)^2 C_{VV} + \rho\sqrt{V}\ S\ \eta(V)\ C_{SV} = r\ C \tag{A.6}$$

where subscripts denote derivatives and the arguments of the *C* function have been suppressed.[16] Next, suppose *f* is a function of *S, V,* and, *t.* The infinitesimal generator of *f(S,V,t)*, denoted $Lf(S,V,t)$, is the function's expected instantaneous rate of change at *t* and is given by (see Hansen and Scheinkman (1995)):

$$Lf(S,V,t) = f_t + \mu(S,V,t)S\,f_S + m(V)\,f_V + \frac{1}{2}VS^2\,f_{SS} + \frac{1}{2}\eta^2(V)\,f_{VV} + \rho\sqrt{V}S\eta(V)\,f_{SV} \tag{A.7}$$

Now let $C^A(S, V, t)$ and $C^B(S, V, t)$ be the price of two call options at date *t* that differ only in that the maturity of option *B* is one month after the maturity of option *A*. Define the function $g(S_1,V_1,t_1)$ as the difference in the returns between the two securities from time *t* to time t_1 (where $t < t_1$):

$$g(S_1,V_1,t_1) \equiv \frac{C^A(S_1,V_1,t_1)}{C^A(S,V,t)} - \frac{C^B(S_1,V_1,t_1)}{C^B(S,V,t)} \tag{A.8}$$

By (A.7), the infinitesimal generator of *g(S,V,t)* is:

$$Lg(S,V,t) = \frac{1}{C^A(S,V,t)}\left[C_t^A + \mu(S,V,t)SC_S^A + m(V)C_V^A + \frac{1}{2}VS^2C_{SS}^A + \frac{1}{2}\eta^2(V)C_{VV}^A + \rho\sqrt{V}S\eta(V)C_{SV}^A\right\rceil$$

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[16] Throughout this outline the context should make it clear whether *t* subscripts are a time index or denote a derivative with respect to time.

$$-\frac{1}{C^B(S,V,t)}\left[C_t^B+\mu(S,V,t)SC_S^B+m(V)C_V^B+\frac{1}{2}VS^2C_{SS}^B+\frac{1}{2}\eta^2(V)C_{VV}^B+\rho\sqrt{V}S\eta(V)C_{SV}^B\right] \quad \text{(A.9)}$$

Substituting in from (A.6) gives:

$$Lg(S,V,t)=\frac{1}{C^A(S,V,t)}\left[\lambda_V\eta(V)C_V^A+S(\mu(S,V,t)-r)C_S^A\right]-\frac{1}{C^B(S,V,t)}\left[\lambda_V\eta(V)C_V^B+S(\mu(S,V,t)-r)C_S^B\right]$$

or, re-arranging:

$$Lg(S,V,t)=\lambda_V\eta(V)\left[\frac{C_V^A(S,V,t)}{C^A(S,V,t)}-\frac{C_V^B(S,V,t)}{C^B(S,V,t)}\right]+\left(\mu(S,V,t)-r\right)\left[\frac{SC_S^A(S,V,t)}{C^A(S,V,t)}-\frac{SC_S^B(S,V,t)}{C^B(S,V,t)}\right] \quad \text{(A.10)}$$

Given suitable restrictions on $\mu(S,V,t)$, $m(V)$, $\eta(V)$, and an otherwise arbitrary function $h(S,V,t)$, the conditional expectation at time t of $h(S_{t+\Delta},V_{t+\Delta},t+\Delta)$ can be written in the form of a Taylor's series expansion:

$$E_t\left[h(S_{t+\Delta},V_{t+\Delta},t+\Delta)\right]=h(S_t,V_t,t)+Lh(S_t,V_t,t)\Delta+\frac{1}{2}L^2h(S_t,V_t,t)\Delta^2+\ldots+\frac{1}{n!}L^nh(S_t,V_t,t)\Delta^n+O(\Delta^{n+1})$$

As in the previous subsection of this Appendix, this implies that a third order approximation to $Lh(S_t,V_t,t)$ is

$$Lh(S_t,V_t,t)=\frac{1}{6\Delta}\left\{18E_t\left[h(S_{t+\Delta},V_{t+\Delta},t+\Delta)-h(S_t,V_t,t)\right]-9E_t\left[h(S_{t+2\Delta},V_{t+2\Delta},t+2\Delta)-h(S_t,V_t,t)\right]+2E_t\left[h(S_{t+3\Delta},V_{t+3\Delta},t+3\Delta)-h(S_t,V_t,t)\right]\right\}+O(\Delta^3)$$

Combining this expression with the definition of $g(S,V,t)$ given in (A.8) yields:

$$Lg(S_t,V_t,t)=\frac{1}{6\Delta}\left\{18E_t\left[R_{t,t+\Delta}^{AB}(S,V,t)\right]-9E_t\left[R_{t,t+2\Delta}^{AB}(S,V,t)\right]+2E_t\left[R_{t,t+3\Delta}^{AB}(S,V,t)\right]\right\}+O\left(\Delta^3\right)$$

where

$$R_{t,t+\tau}^{AB}(S,V,t)\equiv\text{The difference in the returns on calls } A \text{ and } B \text{ between } t \text{ and } t+\tau.$$

Finally, combining this third order expression for the infinitesimal generator with that given in (A.10) and solving for λ_V yields:

$$\lambda_V = \frac{1}{\eta(V)\left[\frac{C_V^A(S,V,t)}{C^A(S,V,t)} - \frac{C_V^B(S,V,t)}{C^B(S,V,t)}\right]}\left\{\frac{1}{6\Delta}\left[18E_t\left[R_{t,t+\Delta}^{AB}(S,V,t)\right] - 9E_t\left[R_{t,t+2\Delta}^{AB}(S,V,t)\right] + 2E_t\left[R_{t,t+3\Delta}^{AB}(S,V,t)\right]\right] - \left[\mu(S,V,t) - r\right]\left[\frac{SC_S^A(S,V,t)}{C^A(S,V,t)} - \frac{SC_S^B(S,V,t)}{C^B(S,V,t)}\right]\right\} + O(\Delta^3) \tag{A.11}$$

Equation (A.11) will be used to estimate the market price of variance risk, λ_V, by constructing nonparametric density estimates of the quantities on the right hand side. Two issues arise when carrying out the estimation. The first issue is that the quantities on the right hand side are functions of *S, V,* and *t* while the market price of variance risk is assumed to depend only on the level of instantaneous variance (see equation (4)). This modeling assumption will be imposed by constructing kernel estimates for the various elements on the right hand side of (A.11) with *V* as the only conditioning variable. Typical elements of the right hand side are then estimated as follows:

$$E_t\left[R_{t,t+\Delta}^{AB}(S,V,t)\right] \approx \frac{\sum_{t=0}^{T-1}\left(\frac{C_{(t+1)\Delta}^A - C_{t\Delta}^A}{C_{t\Delta}^A} - \frac{C_{(t+1)\Delta}^B - C_{t\Delta}^B}{C_{t\Delta}^B}\right)K\left[(V - V_{t\Delta})/h\right]}{\sum_{t=0}^{T-1}K\left[(V - V_{t\Delta})/h\right]}$$

$$\left[\mu(S,V,t) - r\right] \approx \frac{\sum_{t=0}^{T-1}\left(\frac{S_{(t+1)\Delta} - S_{t\Delta}}{S_{t\Delta}} - r_{t\Delta}\right)K\left[(V - V_{t\Delta})/h\right]}{\sum_{t=0}^{T-1}K\left[(V - V_{t\Delta})/h\right]}$$

The second issue that arises when using equation (A.11) to estimate the market price of variance risk is that the derivatives of the prices of the call options *A* and *B* with respect to *S* and *V* are not observable. In principle, these derivatives can be estimated nonparametrically from the observed call prices. However, nonparametric estimation of the derivatives of functions is known to be extremely difficult. This difficulty will be avoided by replacing the C_S/C and C_V/C ratios that appear in equation (A.11) with covariances between option returns and either changes in *S* or *V*. These covariances are more amenable to nonparametric estimation than derivatives of functions.

To see how this replacement will be accomplished recall that by Ito's lemma:

$$\frac{dC}{C} = a(S,V,t)dt + \sqrt{V}S\frac{C_S}{C}dW_S + \eta(V)\frac{C_V}{C}dW_V \tag{A.12}$$

where the details of the $a(S,V,t)$ function are not needed for present purposes. In conjunction with equations (1) and (2) this expression implies that

$$Cov\left(\frac{dC}{C}, dS\right) = \left[VS^2\frac{C_S}{C} + \rho\sqrt{V}S\eta(V)\frac{C_V}{C}\right]dt$$

and

$$Cov\left(\frac{dC}{C}, dV\right) = \left[\rho\sqrt{V}S\eta(V)\frac{C_S}{C} + \eta^2(V)\frac{C_V}{C}\right]dt$$

Denoting the corresponding instantaneous covariances by Cov_{CS} and Cov_{CV}, the preceding two equations can be written in matrix form as:

$$\begin{pmatrix} Cov_{CS} \\ Cov_{CV} \end{pmatrix} = \begin{pmatrix} VS^2 & \rho\sqrt{V}S\eta(V) \\ \rho\sqrt{V}S\eta(V) & \eta^2(V) \end{pmatrix}\begin{pmatrix} C_S/C \\ C_V/C \end{pmatrix}.$$

This matrix is invertible provided that $|\rho| < 1$ which will be assumed. Then

$$\begin{pmatrix} C_S/C \\ C_V/C \end{pmatrix} = \begin{pmatrix} VS^2 & \rho\sqrt{V}S\eta(V) \\ \rho\sqrt{V}S\eta(V) & \eta^2(V) \end{pmatrix}^{-1}\begin{pmatrix} Cov_{CS} \\ Cov_{CV} \end{pmatrix} \tag{A.13}$$

The task then is to estimate Cov_{CS} and Cov_{CV}. In order to do this note that when the drift functions and the diffusion functions of the *S* and *C* processes are denoted, respectively, by μ_s, μ_c, σ_s^2, and σ_C^2, the infinitesimal generator of $f(S_t,C_t)$ is

$$Lf(S_t, C_t) = \mu_s f_S + \mu_c f_C + \frac{1}{2}\sigma_s^2 f_{SS} + \rho^{SC}\sigma_S\sigma_C f_{SC} + \frac{1}{2}\sigma_C^2 f_{CC}.$$ [17]

As a result, the function $j(S_{t_1}, C_{t_1})$ defined by

$$j(S_{t_1}, C_{t_1}) \equiv \left(\frac{C_{t_1} - C_t}{C_t}\right)\left(S_{t_1} - S_t\right) \; where \; t < t_1.$$

has infinitesimal generator

$$Lj(S_{t_1}, C_{t_1}) = \mu_s\left(\frac{C_{t_1} - C_t}{C_t}\right) + \mu_c\left(\frac{S_{t_1} - S_t}{C_t}\right) + \rho^{SC}\sigma_S\sigma_C\frac{1}{C_t}$$

so that

$$Lj(S_t, C_t) = \rho^{SC}\sigma_S\sigma_C\frac{1}{C_t} \tag{A.14}$$

As above, the infinitesimal generator of *j(S*$_t$*, C*$_t$*)* has the third order approximation:

$$L\, j(S_t, C_t) = \frac{1}{6\Delta}\left\{18E_t\left[j(S_{t+\Delta}, C_{t+\Delta}) - j(S_t, C_t)\right] - 9E_t\left[j(S_{t+2\Delta}, C_{t+2\Delta}) - j(S_t, C_t)\right] + 2E_t\left[j(S_{t+3\Delta}, C_{t+3\Delta}) - j(S_t, C_t)\right]\right\} + O(\Delta^3)$$

or

$$L\, j(S_t, C_t) = \frac{1}{6\Delta}\left\{18E_t\left[\left(\frac{C_{t+\Delta} - C_t}{C_t}\right)\left(S_{t+\Delta} - S_t\right)\right] - 9E_t\left[\left(\frac{C_{t+2\Delta} - C_t}{C_t}\right)\left(S_{t+2\Delta} - S_t\right)\right] + 2E_t\left[\left(\frac{C_{t+3\Delta} - C_t}{C_t}\right)\left(S_{t+3\Delta} - S_t\right)\right]\right\} + O(\Delta^3)$$

Combining with equation (A.14) yields

$$Cov_{CS} = \frac{1}{6\Delta}\left\{18E_t\left[\left(\frac{C_{t+\Delta} - C_t}{C_t}\right)\left(S_{t+\Delta} - S_t\right)\right] - 9E_t\left[\left(\frac{C_{t+2\Delta} - C_t}{C_t}\right)\left(S_{t+2\Delta} - S_t\right)\right] + 2E_t\left[\left(\frac{C_{t+3\Delta} - C_t}{C_t}\right)\left(S_{t+3\Delta} - S_t\right)\right]\right\} + O(\Delta^3).$$

A similar argument shows that

$$Cov_{CV} = \frac{1}{6\Delta}\left\{18E_t\left[\left(\frac{C_{t+\Delta} - C_t}{C_t}\right)\left(V_{t+\Delta} - V_t\right)\right] - 9E_t\left[\left(\frac{C_{t+2\Delta} - C_t}{C_t}\right)\left(V_{t+2\Delta} - V_t\right)\right] + 2E_t\left[\left(\frac{C_{t+3\Delta} - C_t}{C_t}\right)\left(V_{t+3\Delta} - V_t\right)\right]\right\} + O(\Delta^3).$$

---

[17] The superscript on ρ indicates that this symbol denotes the correlation between *S* and *C*. Whenever ρ appears without a superscript it denotes the correlation between *S* and *V*.

These covariances will be estimated as a function of the level of the instantaneous variance as follows:

$$Cov_{CS}(V) \approx \frac{\sum_{t=0}^{T-1} \frac{1}{6\Delta}\left\{18*\left[\left(\frac{C_{(t+1)\Delta}-C_{t\Delta}}{C_{t\Delta}}\right)\left(S_{(t+1)\Delta}-S_{t\Delta}\right)\right]-9\left[\left(\frac{C_{(t+2)\Delta}-C_{t\Delta}}{C_{t\Delta}}\right)\left(S_{(t+2)\Delta}-S_{t\Delta}\right)\right]+2\left[\left(\frac{C_{(t+3)\Delta}-C_{t\Delta}}{C_{t\Delta}}\right)\left(S_{(t+3)\Delta}-S_{t\Delta}\right)\right]\right\}K\left[\left(V-V_{t\Delta}\right)/h\right]}{\sum_{t=0}^{T-1} K\left[\left(V-V_{t\Delta}\right)/h\right]}$$

and

$$Cov_{CV}(V) \approx \frac{\sum_{t=0}^{T-1} \frac{1}{6\Delta}\left\{18*\left[\left(\frac{C_{(t+1)\Delta}-C_{t\Delta}}{C_{t\Delta}}\right)\left(V_{(t+1)\Delta}-V_{t\Delta}\right)\right]-9\left[\left(\frac{C_{(t+2)\Delta}-C_{t\Delta}}{C_{t\Delta}}\right)\left(V_{(t+2)\Delta}-V_{t\Delta}\right)\right]+2\left[\left(\frac{C_{(t+3)\Delta}-C_{t\Delta}}{C_{t\Delta}}\right)\left(V_{(t+3)\Delta}-V_{t\Delta}\right)\right]\right\}K\left[\left(V-V_{t\Delta}\right)/h\right]}{\sum_{t=0}^{T-1} K\left[\left(V-V_{t\Delta}\right)/h\right]}.$$

Define *M* as the matrix which appears in equation (A.13):

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \equiv \begin{pmatrix} VS^2 & \rho\sqrt{V}S\eta(V) \\ \rho\sqrt{V}S\eta(V) & \eta^2(V) \end{pmatrix}^{-1}$$

The elements of *M* will then be estimated as a function of the variance level by the following expressions:

$$M_{11}(V) \approx \frac{\sum_{t=0}^{T} \left(V_{t\Delta}S_{t\Delta}^2\right)K\left[\left(V-V_{t\Delta}\right)/h\right]}{\sum_{t=0}^{T} K\left[\left(V-V_{t\Delta}\right)/h\right]}$$

$$M_{12}(V) = M_{21}(V) \approx \frac{\sum_{t=0}^{T} \rho\sqrt{V_{t\Delta}}S_{t\Delta}\eta\left(V_{t\Delta}\right)K\left[\left(V-V_{t\Delta}\right)/h\right]}{\sum_{t=0}^{T} K\left[\left(V-V_{t\Delta}\right)/h\right]}$$

$$M_{22}(V) \approx \frac{\sum_{t=0}^{T} \left(\eta^2\left(V_{t\Delta}\right)\right)K\left[\left(V-V_{t\Delta}\right)/h\right]}{\sum_{t=0}^{T} K\left[\left(V-V_{t\Delta}\right)/h\right]}.$$

Finally, making use of equation (A.13) the derivative terms in equation (A.11) will be estimated as functions of the level of instantaneous variance by:

$$\frac{C_S}{C}(V) = M_{11}(V)Cov_{CS}(V) + M_{12}(V)Cov_{CV}(V)$$

and

$$\frac{C_V}{C}(V) = M_{21}(V)Cov_{CS}(V) + M_{22}(V)Cov_{CV}(V).$$

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Table I
Descriptive Statistics for S&P 500 Index Options Data, June 1, 1988 through December 29, 1995

Descriptive statistics for the sample of CBOE traded S&P500 index calls and puts from the period June 1, 1988 through December 29, 1995. Panel A reports on the two put-call parity pairs on each trade date with shortest maturity greater than nine calendar days that are closest to ATM. Panel B reports on the two put-call parity pairs on each trade date with second to shortest maturity greater than nine calendar days that are closest to ATM. Panel C reports on the single put-call parity pair on each trade date with shortest maturity greater than nine calendar days that is closest to ATM.

Variable	Mean	S.D.	Min	Percentiles					Max
				5%	10%	50%	90%	95%	
Panel A: Two Put-Call Parity Pairs with Shortest Maturity Greater than Nine Calendar Days which are Closest to ATM									
Call Price ($)	5.62	2.21	0.13	2.25	2.88	5.50	8.50	9.38	18.75
Put Price ($)	5.54	1.98	1.13	2.50	3.00	5.50	8.13	9.00	20.63
BS Implied Vol (%)	0.137	0.039	0.067	0.091	0.095	0.132	0.191	0.216	0.362
BS Implied Var (%^{2})	0.0204	0.0128	0.0045	0.0083	0.0091	0.0174	0.0365	0.0467	0.1311
T (Calendar Days)	25.18	9.87	10	11	12	25	37	40	72
K (Index Points)	404.05	81.75	255	275	295	410	500	560	625
r (%)	0.052	0.018	0.024	0.028	0.029	0.053	0.077	0.081	0.097
moneyness	1.000	0.008	0.960	0.988	0.990	1.000	1.010	1.012	1.074
Panel B: Two Put-Call Parity Pairs with Second to Shortest Maturity Greater than Nine Calendar Days which are Closest to ATM									
Call Price ($)	9.47	4.07	0.63	4.88	5.63	9.00	13.63	15.75	71.00
Put Price ($)	9.01	3.02	0.75	5.13	5.88	8.63	12.50	14.25	37.63
BS Implied Vol	0.145	0.040	0.068	0.097	0.102	0.140	0.203	0.226	0.329
BS Implied Var	0.0227	0.0136	0.0046	0.0094	0.0104	0.0197	0.0412	0.0511	0.1084
T (Calendar Days)	60.57	20.76	38.00	39.00	43.00	54.00	93.00	120.00	135.00
K (Index Points)	406.15	81.79	230	275	300	410	505	565	630
r (%)	0.054	0.019	0.027	0.029	0.030	0.054	0.080	0.083	0.092
moneyness	0.999	0.016	0.796	0.980	0.987	0.999	1.011	1.016	1.139
Panel C: Put-Call Parity Pair with Shortest Maturity Greater than Nine Calendar Days which is Closest to ATM									
Call Price ($)	5.29	1.64	1.31	2.88	3.25	5.13	7.50	8.13	13.00
Call Price Next ($)	5.36	2.22	0.63	2.25	2.75	5.00	8.13	9.50	19.00
Put Price ($)	5.23	1.62	1.38	2.75	3.25	5.13	7.25	8.00	13.25
BS Implied Vol	0.133	0.038	0.067	0.091	0.095	0.126	0.182	0.211	0.312
BS Implied Var	0.0191	0.0124	0.0045	0.0083	0.0090	0.0158	0.0333	0.0443	0.0970
T (Calendar Days)	23.24	8.82	10.00	10.00	11.00	23.00	36.00	38.00	45.00
K (Index Points)	415.75	81.33	255	280	305	420	520	565	625
r (%)	0.050	0.018	0.024	0.028	0.028	0.050	0.077	0.080	0.097
moneyness	1.000	0.005	0.980	0.993	0.995	1.000	1.005	1.007	1.020

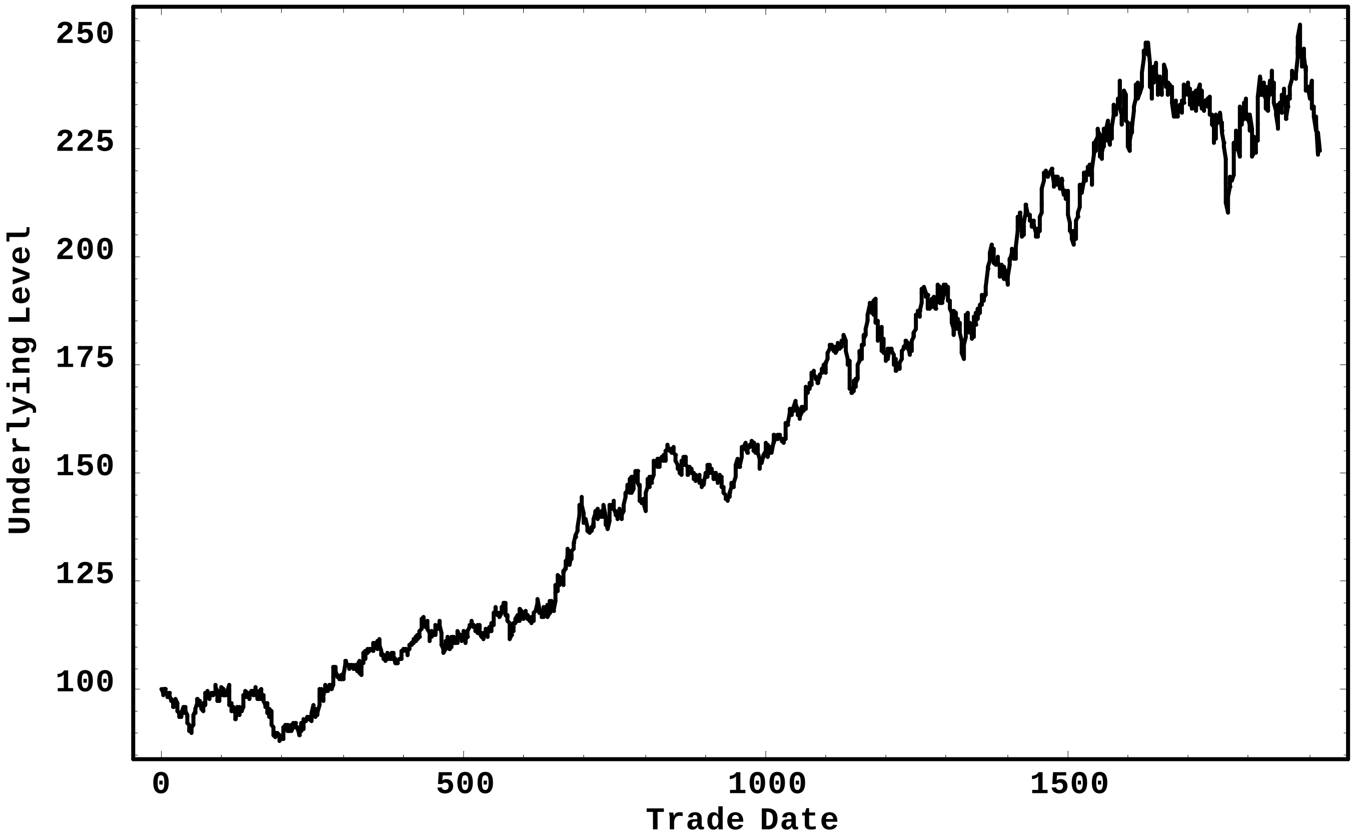

Figure 1. Simulated underlying level.

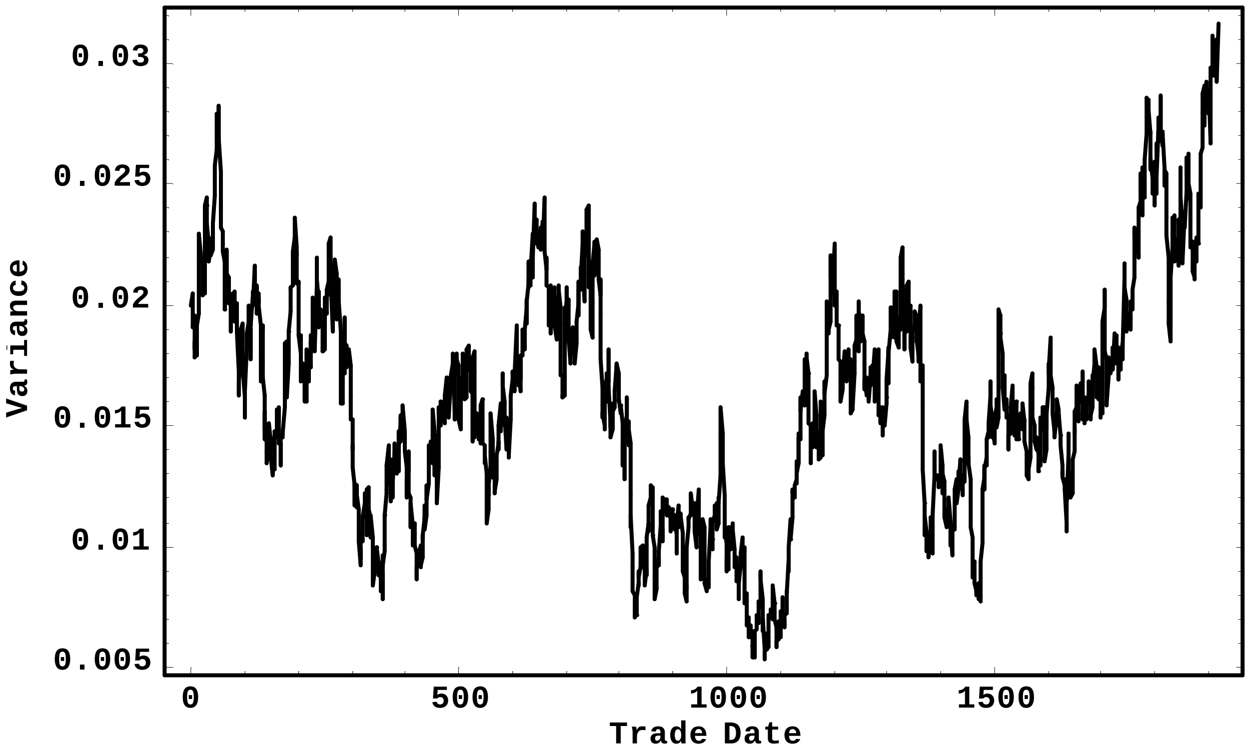

Figure 2. Simulated variance.

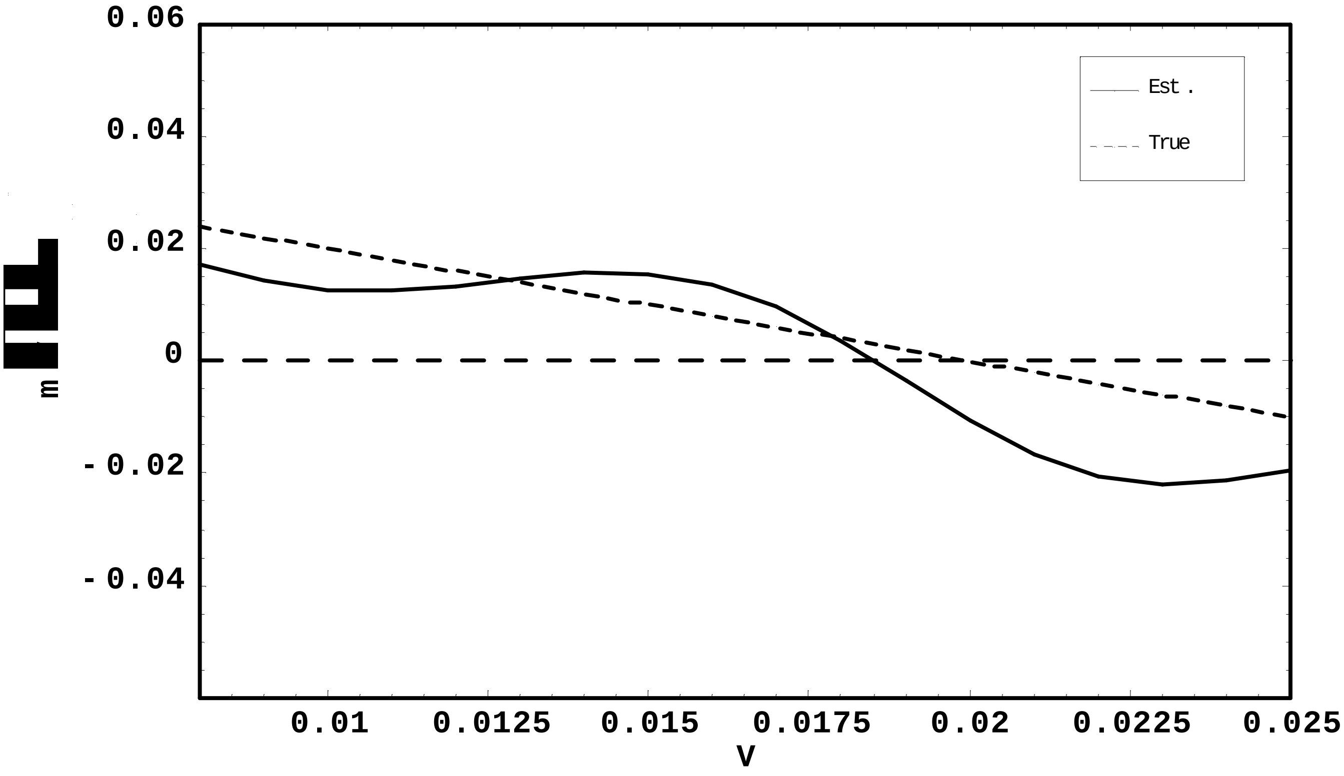

Figure 3. Drift of variance process estimated from artificial data.

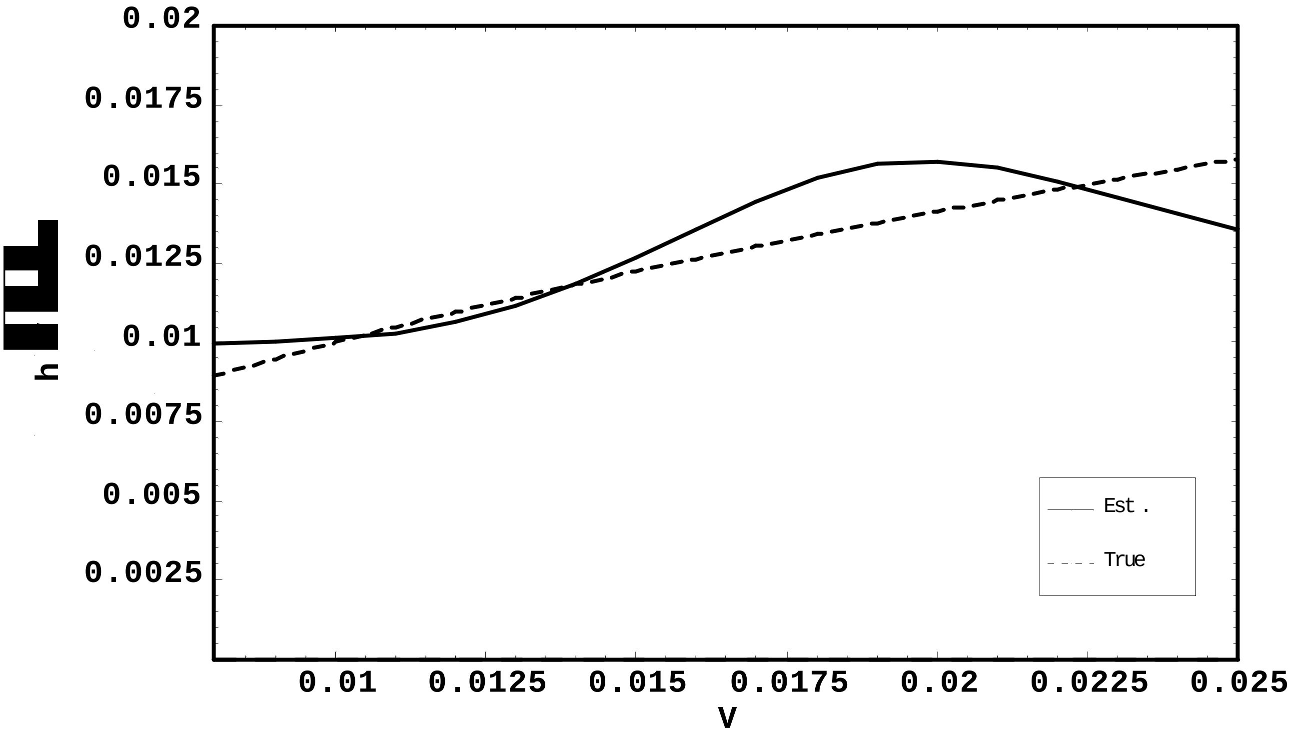

Figure 4. Diffusion of variance process estimated from artificial data.

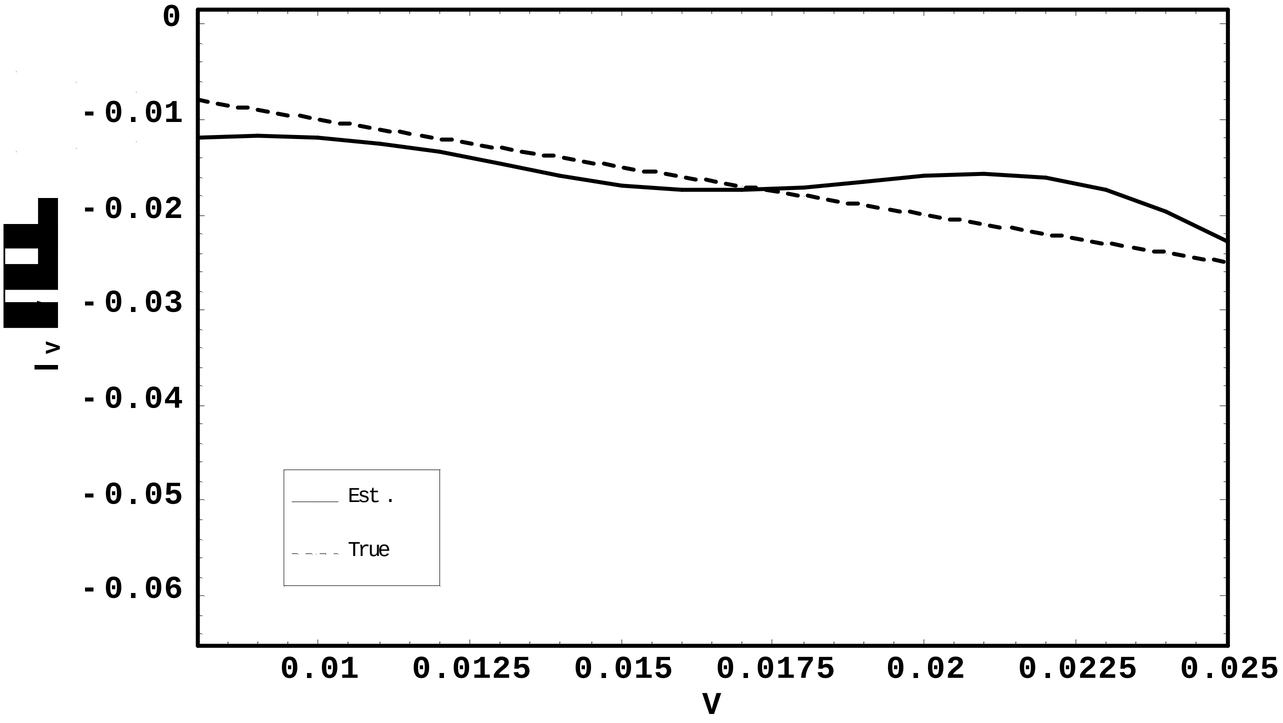

Figure 5. Market price of variance risk estimated from artificial data.

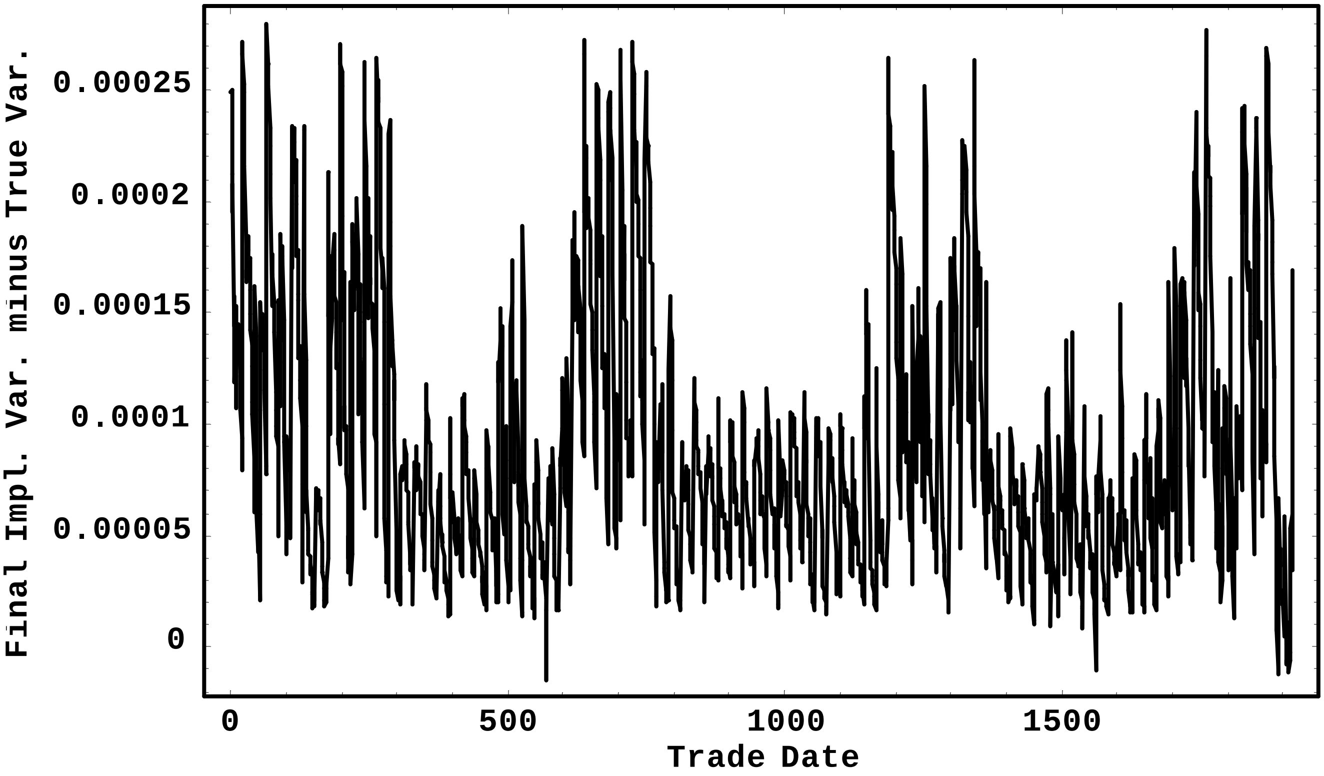

Figure 6. Difference between final iteration implied variances and true variances.

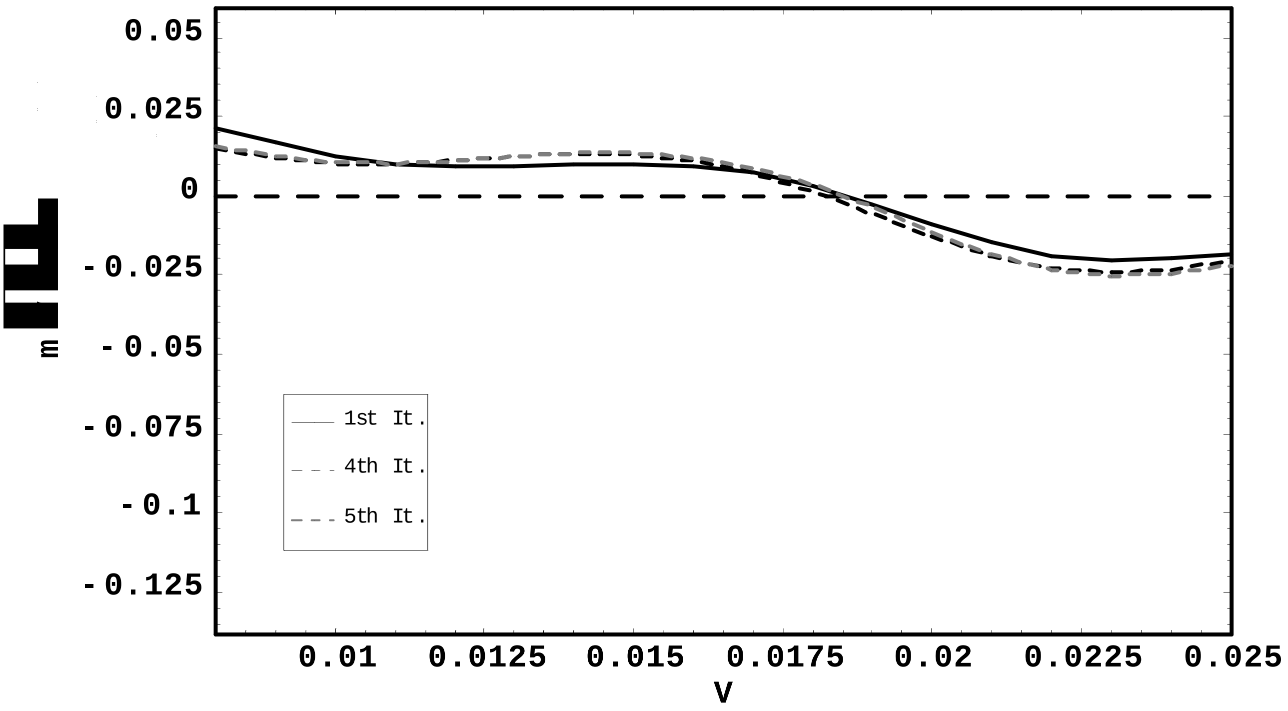

Figure 7. Drift of variance process estimated from artificial data on the 1st, 4th, and 5th iterations of the EM-type algorithm.

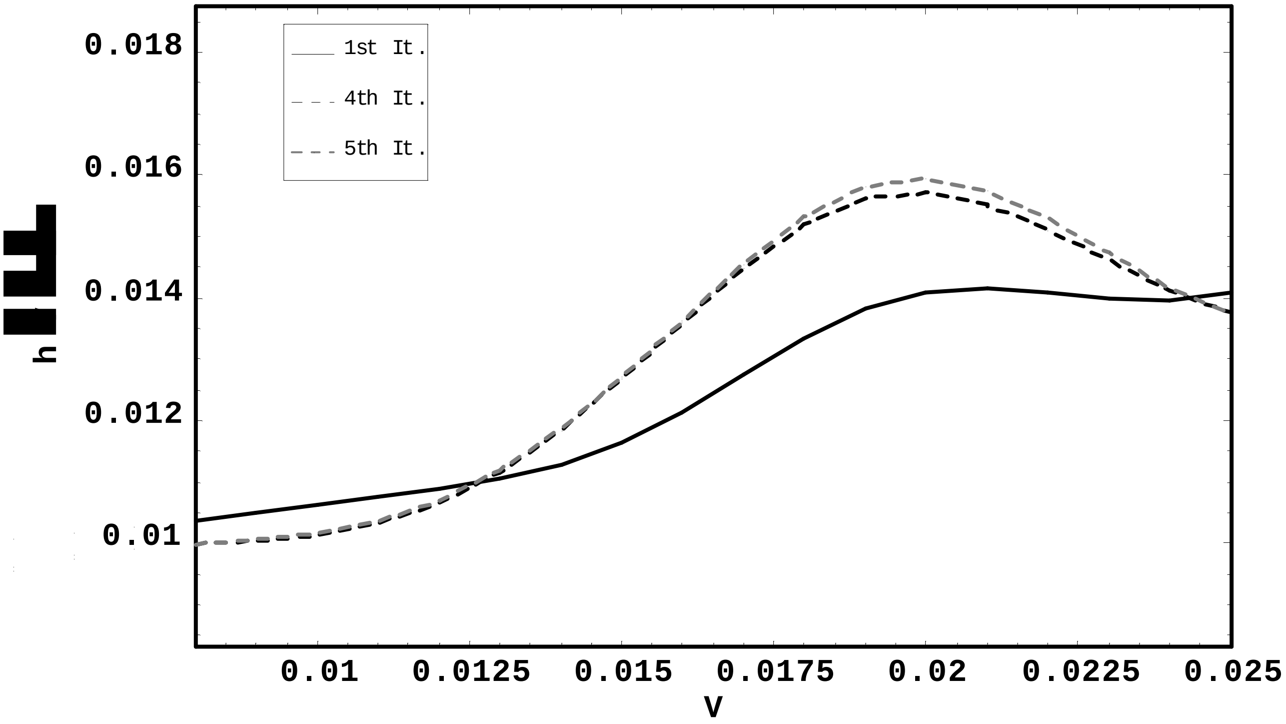

Figure 8. Diffusion of variance process estimated from artificial data on the 1st, 4th, and 5th iterations of the EM-type algorithm.

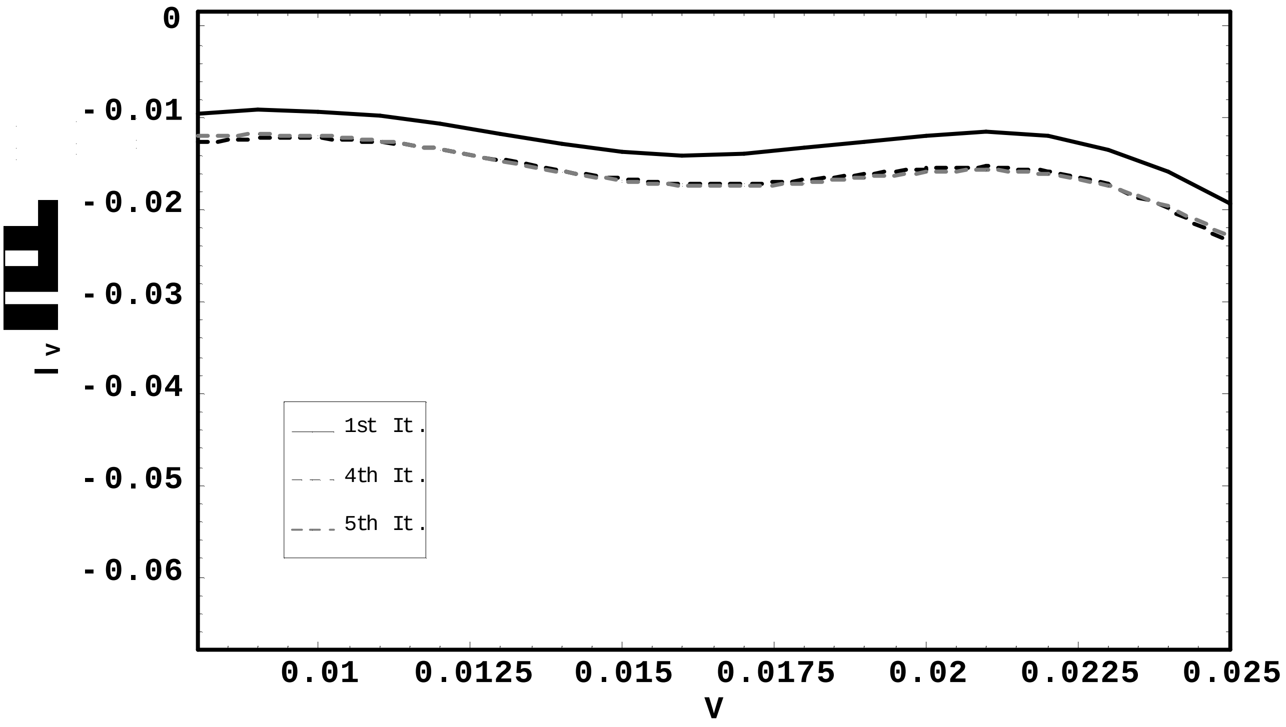

Figure 9. Market price of variance risk estimated from artificial data on the 1st, 4th, and 5th iterations of the EM-type algorithm.

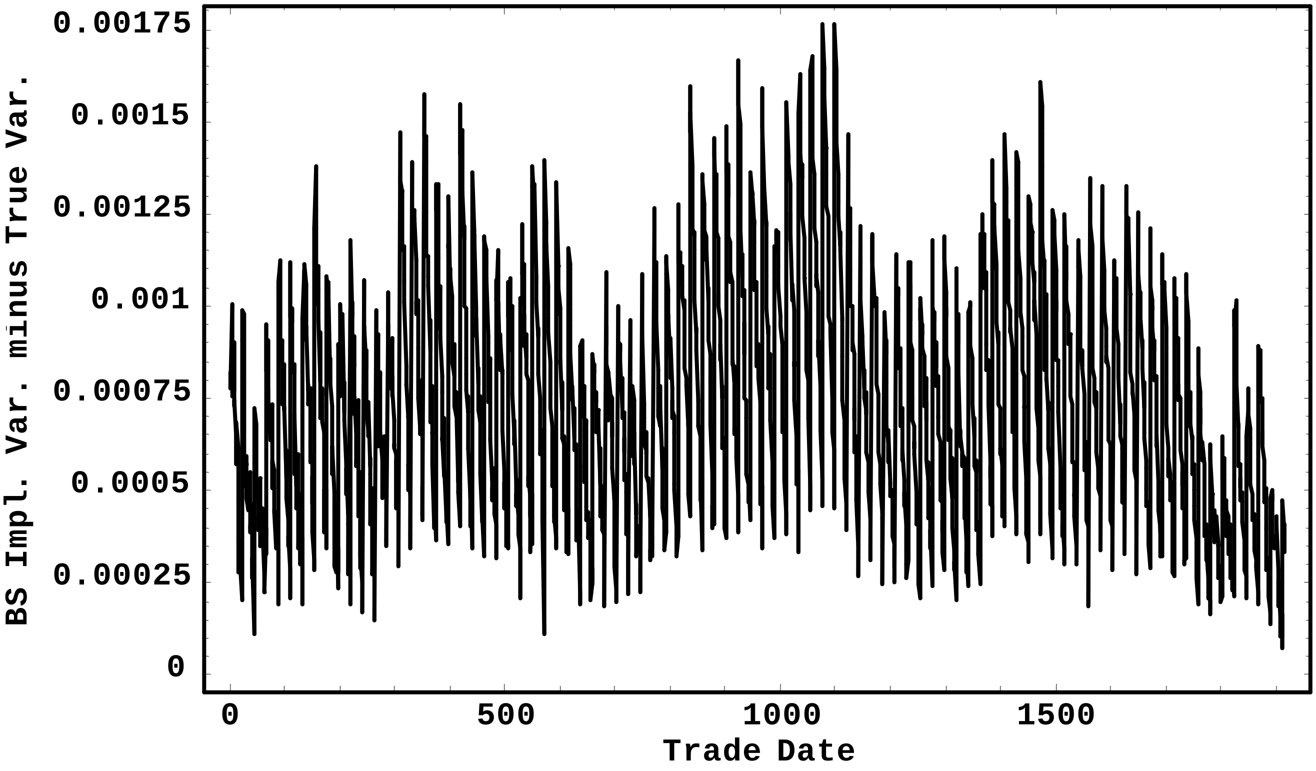

Figure 10. Difference between Black-Scholes implied variances and true variances.

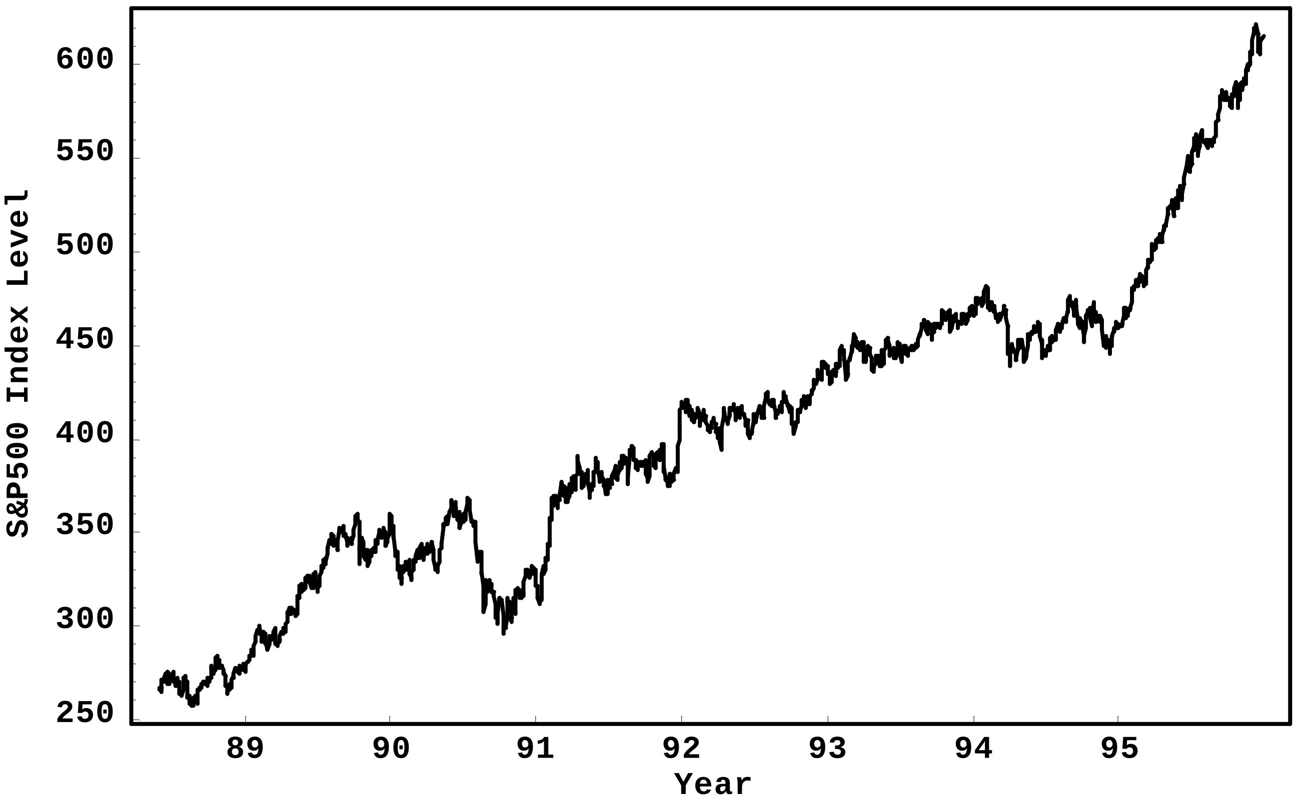

Figure 11. S&P 500 index level.

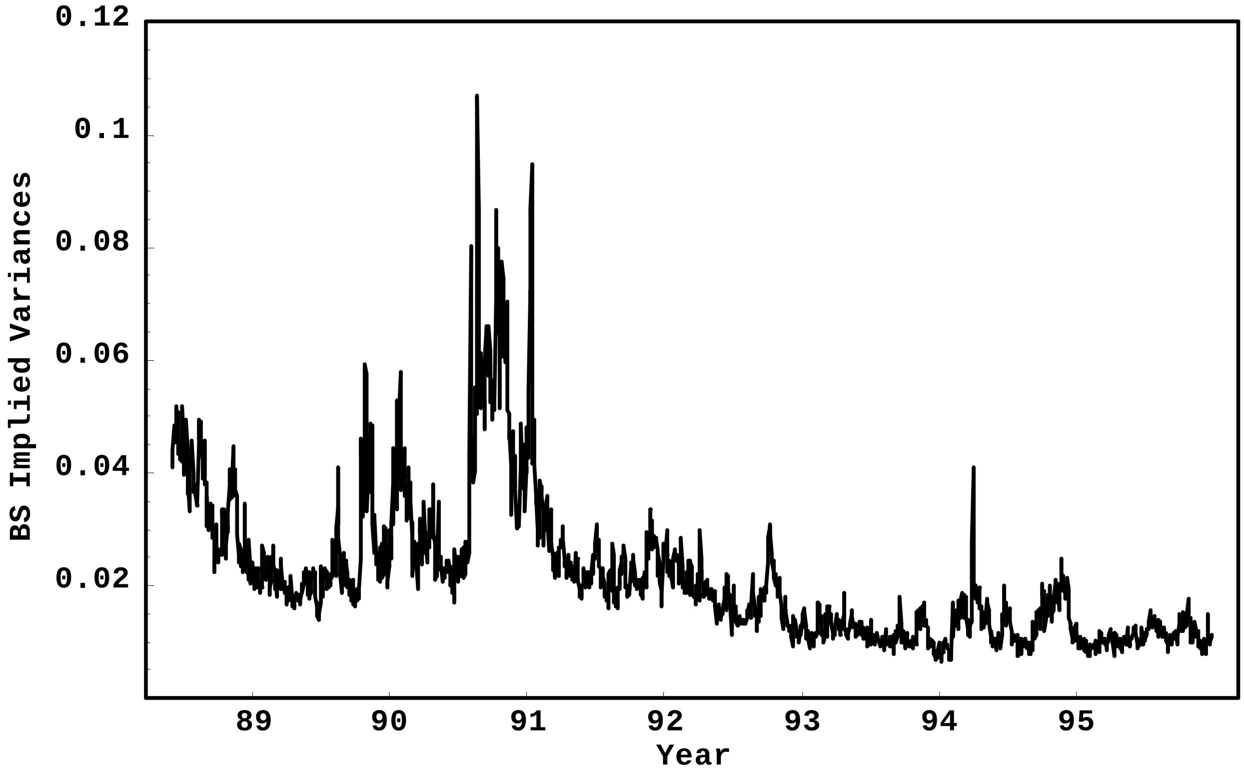

Figure 12. Black-Scholes implied variances from short maturity close to ATM calls.

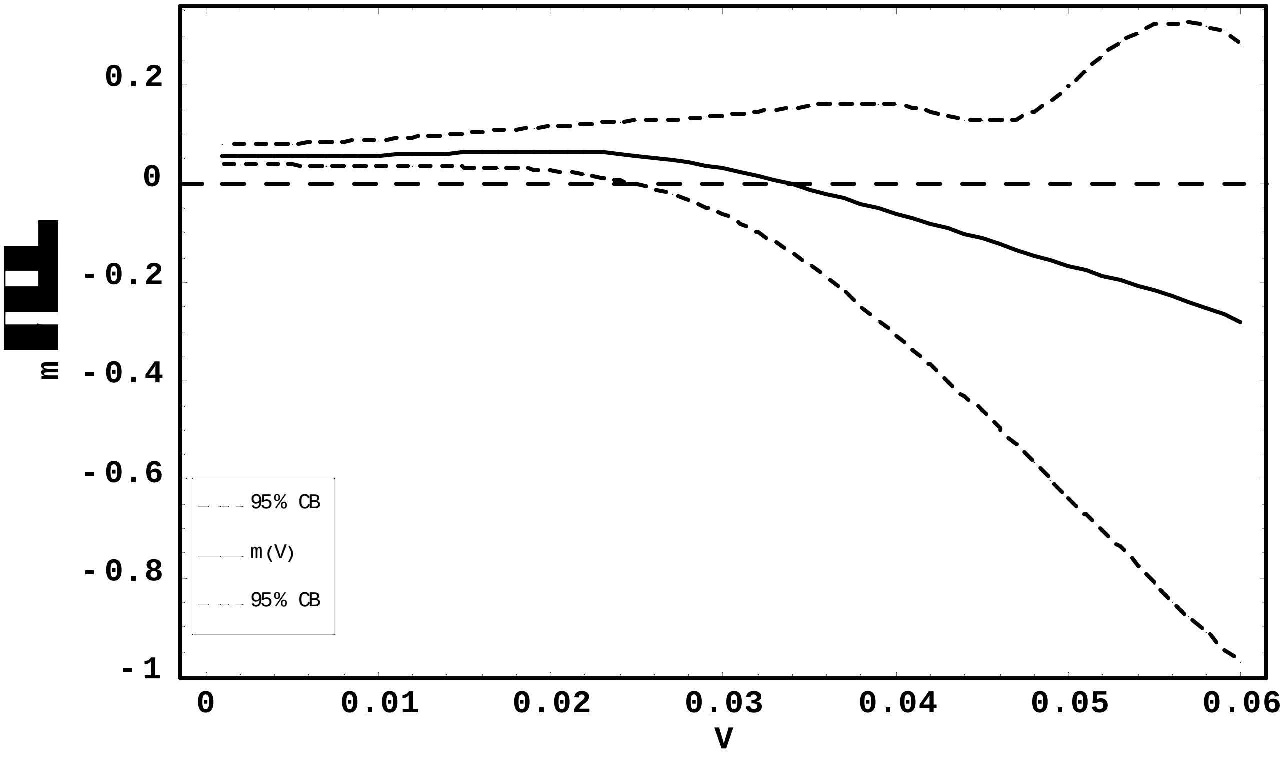

Figure 13. Drift of variance process estimated from S&P 500 index options, June 1, 1988 through December 29, 1995.

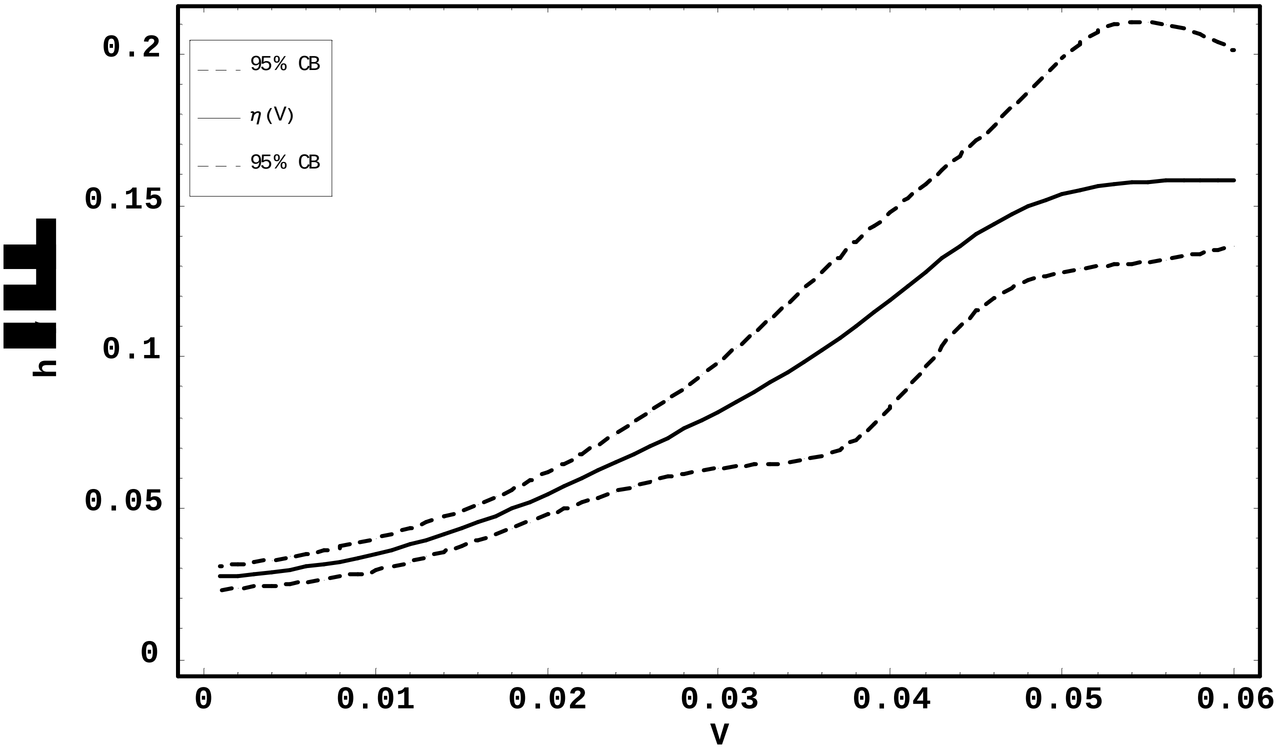

Figure 14. Diffusion of variance process estimated from S&P 500 index options, June 1, 1988 through December 29, 1995.

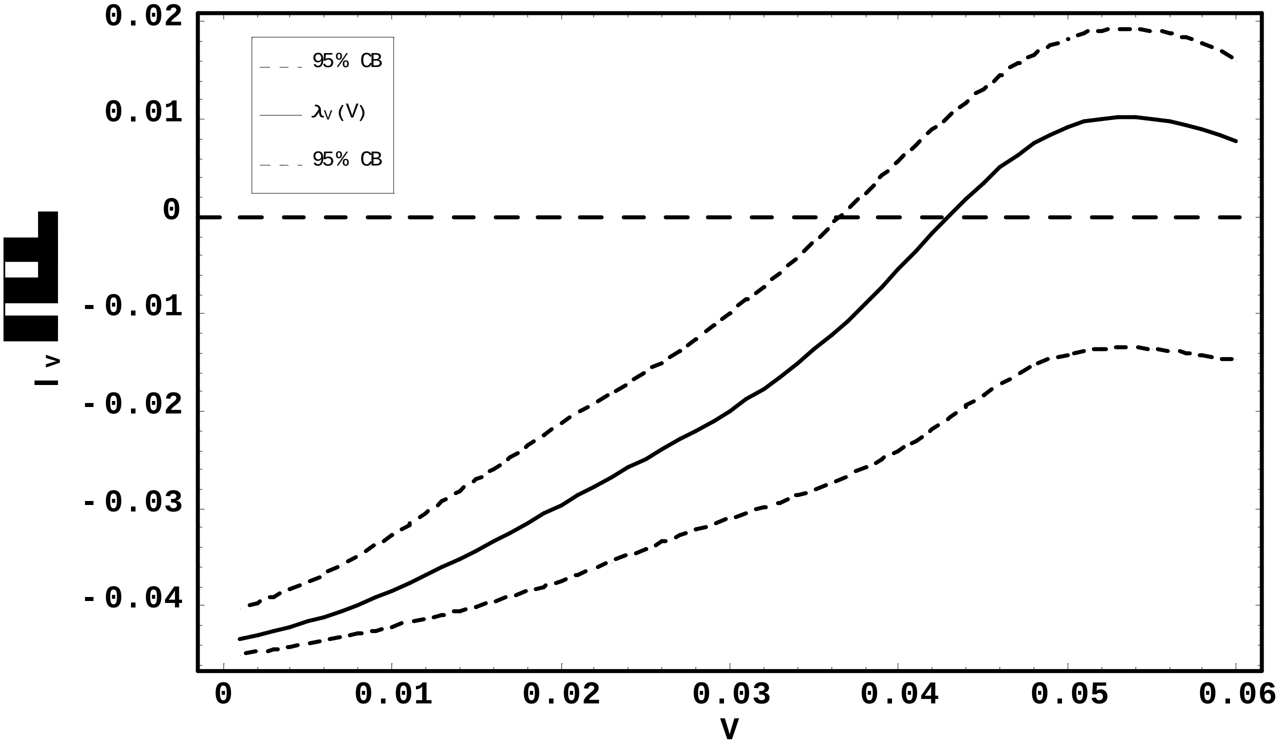

Figure 15. Market price of variance risk estimated from S&P 500 index options, June 1, 1988 through December 29, 1995.

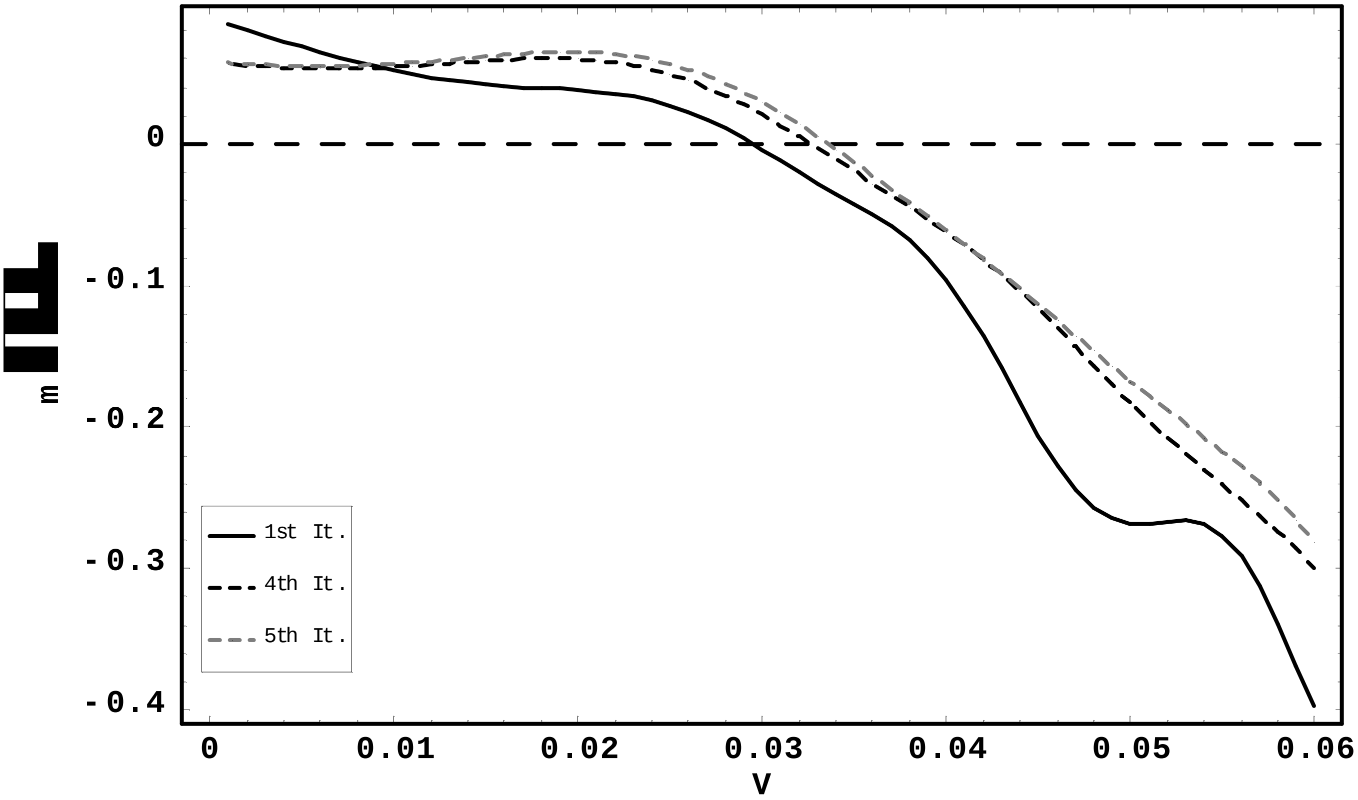

Figure 16. Drift of variance process estimated from S&P 500 index options on the 1st, 4th, and 5th iterations of the EM-type algorithm.

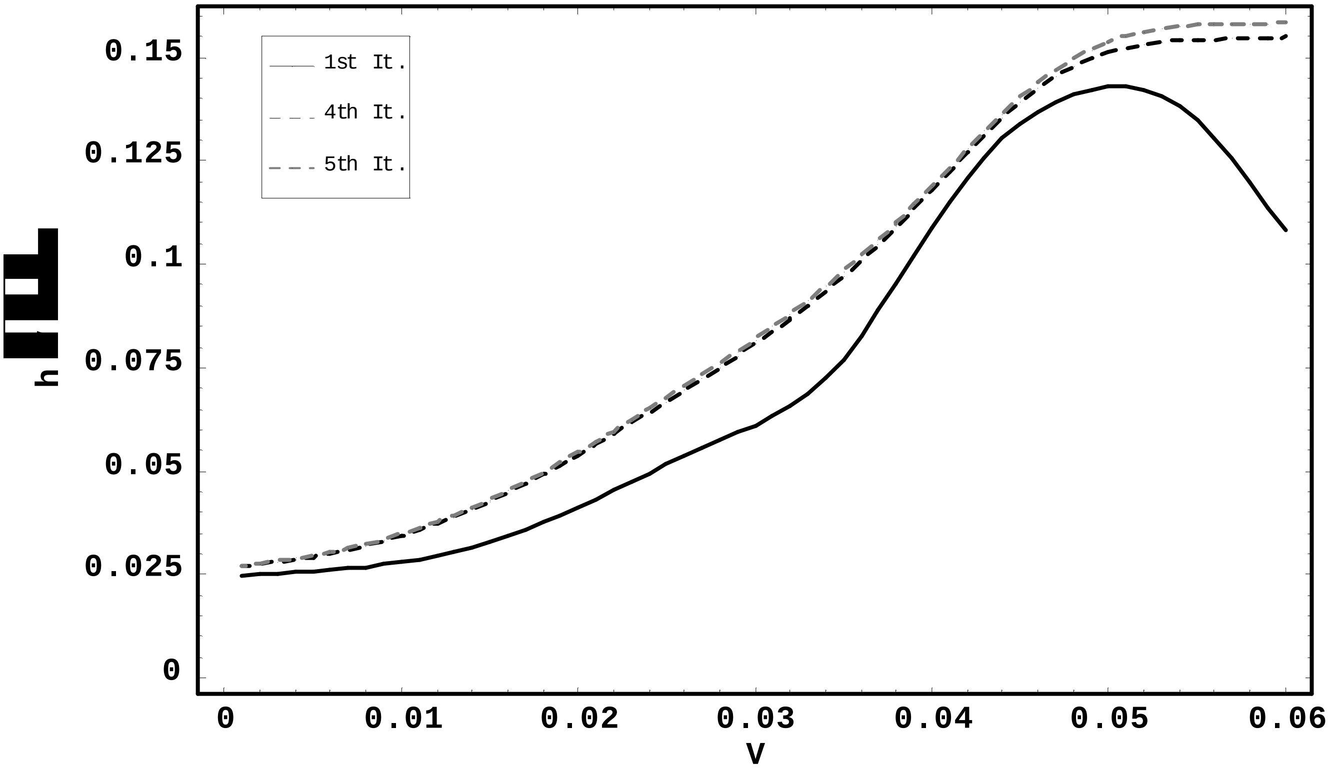

Figure 17. Diffusion of variance process estimated from S&P 500 index options on the 1st, 4th, and 5th iterations of the EM-type algorithm.

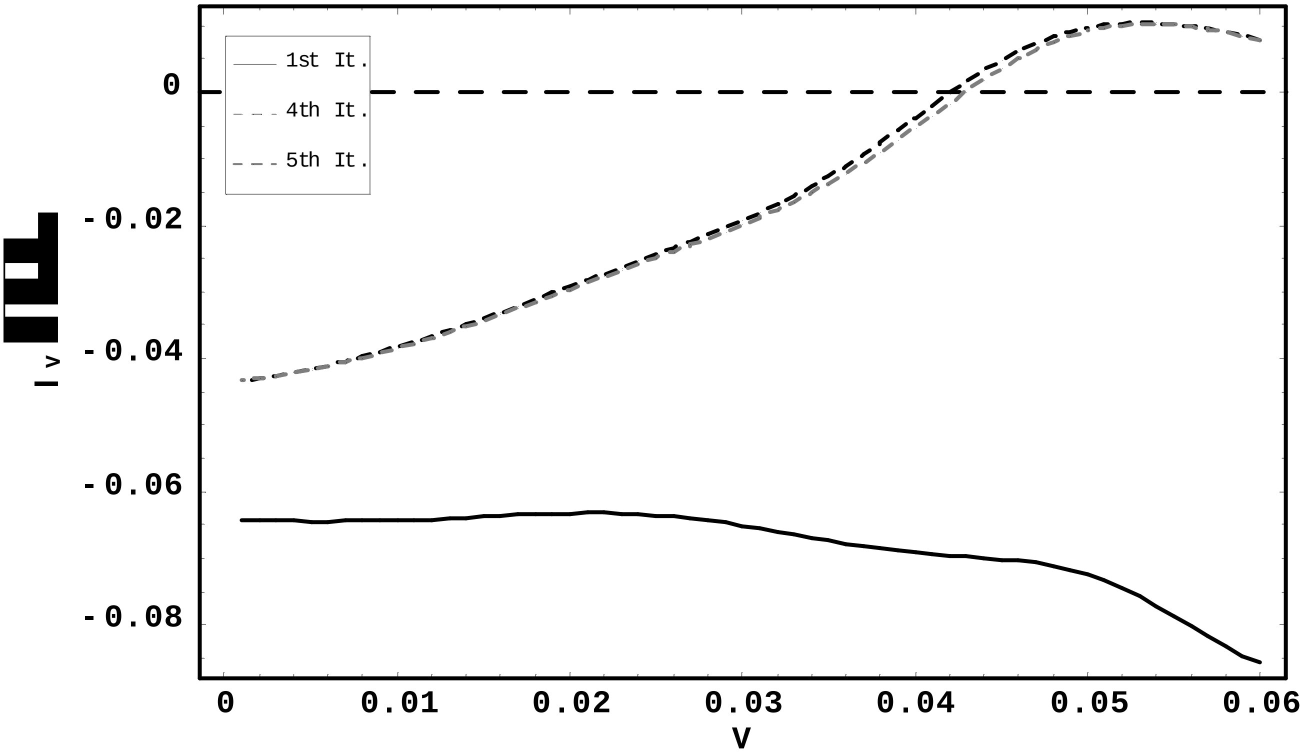

Figure 18. Market price of variance risk estimated from S&P 500 index options on the 1st, 4th, and 5th iterations of the EM-type algorithm.

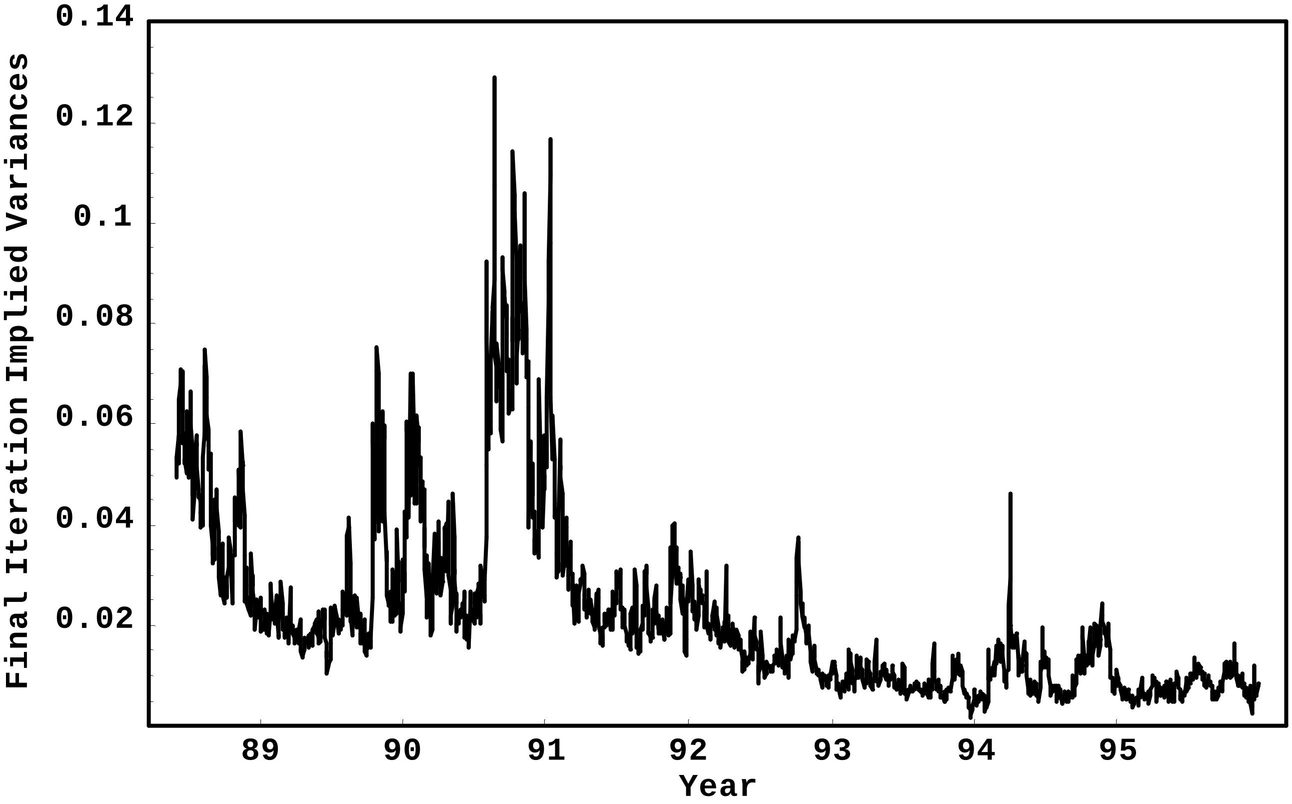

Figure 19. Final iteration implied instantaneous variances.

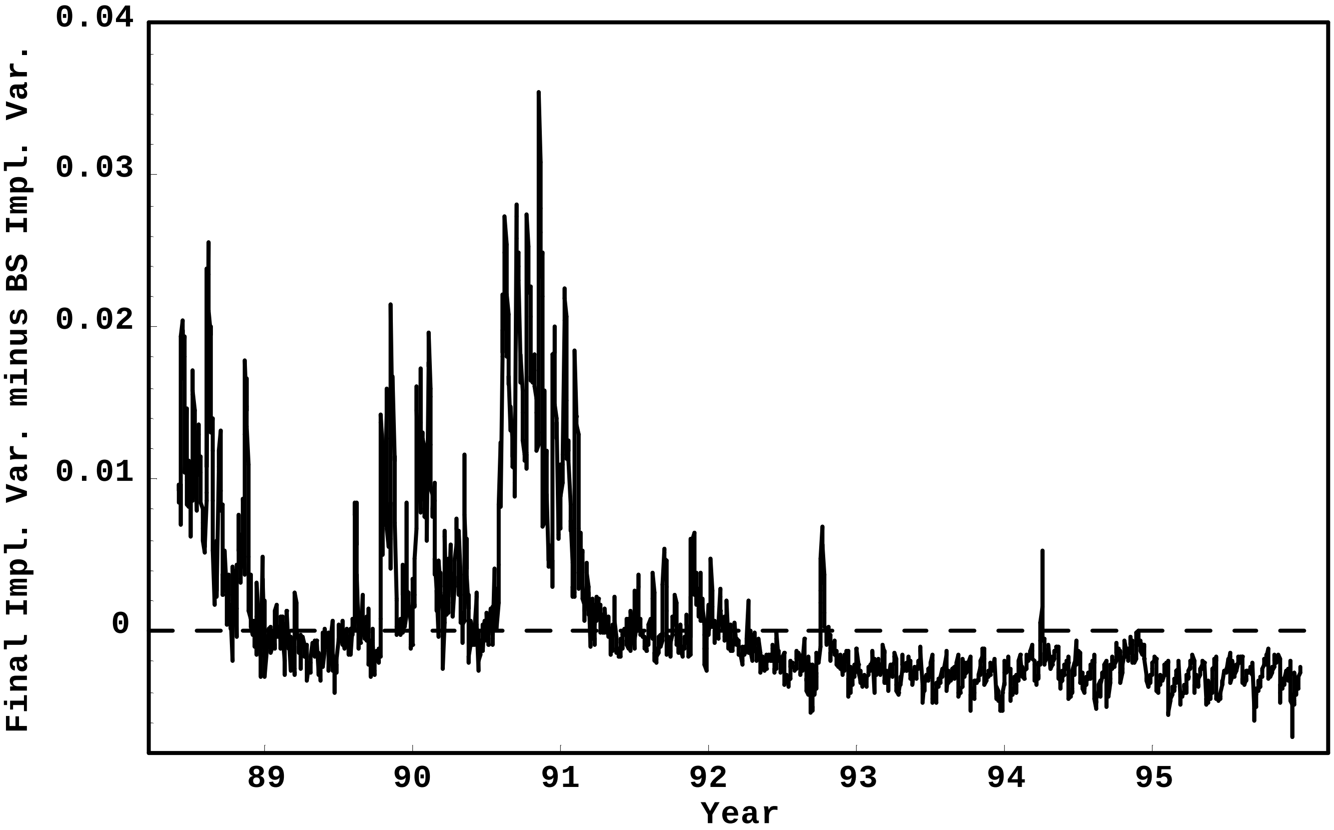

Figure 20. Difference between final iteration implied variances and Black-Scholes implied variances.